# Insider versus outsider information: An asset allocation perspective*

Simon Lysbjerg Hansen†

July 7, 2008

**Abstract**

Information is crucial for decision makers. The more and the better information one possesses, the more qualified a decision one is able to make. In a partial equilibrium framework with incomplete information about the expected stock return, the effect of having private information about either the expected stock return (outsider information) or the terminal stock price (insider information) is studied. A condition for the two types of information to give the same results is provided.

*This paper is a revised version of Hansen (2006, Chapter 2). I would like to thank the members of my Ph. D. committee and in particular Holger Kraft for his many constructive comments.

†Department of Business and Economics, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark. E-mail: slh@sam.sdu.dk.

# 1 Introduction

The problem of choosing optimal investment and consumption strategies has been widely studied. In continuous time theory the pioneering work conducted by Merton (1969, 1971) are standard references. Since then Merton's framework has been applied to many interesting financial problems to capture empirically observed investment and consumption behavior. One line of research, to which this paper also belongs, studies the effect of having incomplete information about the return distribution of the traded securities. The classic papers are Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) whom were among the first to apply the mathematical field of filtering theory to financial economics. In their work, the asset pricing implications of having incomplete information were studied.

Assuming Gaussian uncertainty about the mean rate of return on a risky asset, Brennan (1998) applies the dynamic programming approach and studies effects on optimal portfolio choice by solving the Hamilton-Jacobi-Bellman equation numerically. Lakner (1998) applied the martingale approach and provided integral representations of the optimal portfolio strategy to study the same problem. Apparently, a closed-form solution to the portfolio choice problem was first derived by Rogers (2001) for the single asset case and by Cvitanić, Lazrak, Martellini, and Zapatero (2006) for the case with an arbitrary number of assets. Rogers (2001) compares the effect of parameter uncertainty with the effect of infrequent trading. In the case of parameter uncertainty, the effect is measured relative to an investor who gets a perfect signal about the unknown parameter, which can be interpreted as the value of having full information disclosure. Cvitanić et al. (2006) apply their closed-form solution to test the usefulness of investing optimally in a market with many stocks compared to a naive investment strategy. Further, an analysis of analysts' recommendations is carried out by comparing the utility derived from following such recommendations to the utility derived from being restricted to investing in the market portfolio and the risk-free asset. Other financial applications of parameter uncertainty include Brennan and Xia (2001) who study the effect of being able to learn, whether an observed asset pricing anomaly is genuine or not, and Xia (2001) who examines the effects of uncertainty about the stock return predictability on optimal dynamic portfolio choice. Like in the previous mentioned papers, the incomplete information studied here will also be Gaussian, but other modeling approaches exist. For example, David (1997), Veronesi (1999), and Honda (2003) model incomplete information through hidden Markov chains.

Whereas incomplete information shrinks the information set available to the investor another branch in the literature has focused on the so-called enlargement of filtration, a field initiated by Itô and further developed by Yor, Jeulin, and Jacod. The idea is to expand the information set of an agent with a signal about the future value of some random quantity. A financial investor will take this extra information into account, when the portfolio choice is made. In other words, the investor will try to anticipate the future from the new knowledge available to him. Pikovsky and Karatzas (1996) used the initial enlargement of filtration technique to study how a logarithmic investor behaves under various enlargements. The value of having access to the enlarged filtration is studied by Amendinger, Imkeller, and Schweizer (1998) for a logarithmic investor. The value of information for an investor who derives power or exponential utility is studied in Amendinger, Becherer, and Schweizer (2003), but the explicit calculation in the case of power utility is omitted in the example. Liu, Peleg, and Subrahmanyam (2005) derives the value of private information for the power utility case, and study the effects of having long-lived private information on portfolio choice, consumption, and utility gain.

In the existing literature, the present paper is closest in spirit to the work of Rogers (2001) and Liu et al. (2005), but with the following differences. Whereas Rogers (2001) only consider the value of a perfect signal about the expected rate of return, this study will analyze the effect of varying the precision of a *noisy* signal. In the analysis of Liu et al. (2005) there is complete information about the expected rate of return. However, to be able to compare the two types of signal, the point of departure should be the same. Hence, it is necessary to apply the enlargement of filtration technique to a setting with *incomplete* information. Such a framework has been suggested by Benarous, Mazliak, and Rouge (2000), but in much more generality than is needed for the application in this paper. A seemingly more appealing approach to the modeling of financial markets with incomplete and private information is therefore suggested. For financial economists the approach will probably be easier to grasp. The final contribution of the paper is an information equivalence result, that states a condition under which an investor is indifferent between receiving insider or outsider information.

To illustrate the idea in this paper, consider a Black-Scholes market in which the stock price follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $\mu$ and $B_t$ are unobservable and hence must be estimated by the investor from observations of the stock price. To make more qualified investment decisions investors will in general attach a value to information that is correlated with one of the unobservable quantities. The purpose of this project is to study the effects of information about

1. the drift term through a signal about the expected stock return, $\mu$, and
2. the diffusion term through a signal about the terminal stock price which, in a Black-Scholes setting, is equivalent to a signal about the underlying Brownian motion, $B_T$.

As the precision of the two types of signals tends to infinity, one important difference between the two types of signal is, that the utility obtainable from having the first signal tends to the utility in a model with complete information whereas the utility obtainable from the second signal tends to infinity.

## 2 The mathematical model

Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ on which a one-dimensional Brownian motion $(B_t)$ is defined. The complete information filtration $\mathcal{F}_t$ is the augmentation of the filtration $\mathcal{F}_t^B \times \mathcal{H}$, where $\mathcal{F}_t^B \triangleq \sigma(B_u \,|\, 0 \le u \le t)$ is the filtration generated by the Brownian motion and $\mathcal{H}$ is a $\sigma$-field independent of $\mathcal{F}_t^B$ allowing for parameter uncertainty.

Consider a simple financial market in which two assets are traded. The first asset is risk-free and provides a constant continuously compounded rate of return, $r$. The second asset is risky and will often be referred to as the stock. The stock price is described by the stochastic differential equation

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \\ S_0 &= 1, \end{aligned}$$

where $\mu$ is the instantaneous mean rate of return and $\sigma$ is the instantaneous volatility. Economic agents face incomplete information in the sense that $r$, $\sigma$ and the stock price process $(S_t)$ are observable, whereas $\mu$ and $(B_t)$ are unobservable and must be estimated from observations of the stock price process. Since the volatility can be inferred from the quadratic variation process $\langle S, S\rangle_t$, the assumption that $\sigma$ is observable is justified. From an econometric point of view the problem of estimating expected returns is also much harder than estimating volatility, see e.g. Merton (1980). Instead of working with the mean rate of return, $\mu$, it turns out to be convenient to introduce the market price of risk, $\theta \triangleq \frac{\mu - r}{\sigma}$, and treat this as the unobservable quantity. It is assumed that $\theta \sim \mathcal{N}(m, v)$ and independent of $B_t$, i.e. $\theta$ is $\mathcal{H}$-measurable. In the usual filtering jargon the problem is to estimate the system $\theta$ from observations of the stock price, i.e.

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observation} & dS_t = (r + \sigma\theta) S_t dt + \sigma S_t dB_t & S_0 = 1. \end{array}$$

The first step is to transform the above filtering problem to the linear Kalman-Bucy filter setting. Introducing the process $\bar{B}_t \triangleq B_t + \theta t$, the dynamics of the stock price can be rewritten in terms of $\bar{B}_t$ as

$$\begin{aligned} dS_t &= r S_t dt + \sigma S_t d\bar{B}_t \\ S_0 &= 1, \end{aligned}$$

which has the solution

$$S_t = \exp\left(\left[r - \frac{\sigma^2}{2}\right] t + \sigma \bar{B}_t\right).$$

From this, it is clear that $\bar{B}_t$ and $S_t$ generate the same information, i.e. $\mathcal{F}_t^S = \bar{\mathcal{F}}_t$ where $\mathcal{F}_t^S \triangleq \sigma(S_u \,|\, 0 \leq u \leq t)$ and $\bar{\mathcal{F}}_t \triangleq \sigma(\bar{B}_u \,|\, 0 \leq u \leq t)$. Hence the filtering problem can be formulated as

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observation} & d\bar{B}_t = \theta dt + dB_t & \bar{B}_0 = 0. \end{array}$$

The financial market faced by an investor depends on the information available to him. The mathematical object used to model information is the concept of a $\sigma$-algebra. As described above the public information is generated by the stock price, $\mathcal{F}_t^S = \bar{\mathcal{F}}_t$, whereas the information conveyed by the underlying Brownian motion, $\mathcal{F}_t^B$, is inaccessible to the investor. Furthermore, the investor might receive a signal, $\psi$, at time $t = 0$ about either the expected stock return (through $\theta$) or the future stock price (through $\bar{B}_T$). Since the signal is observable, the filtering problem becomes

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observation} & d\bar{B}_t = \theta dt + dB_t & \bar{B}_0 = 0 \\ & d\psi = 0 & \psi \in \{\psi_\theta, \bar{\psi}\}, \end{array}$$

in which the signal is left unspecified for now since the dependence structure between $\psi$ and $\mathcal{F}_t^B$ will determine whether $B_t$ is a Brownian motion in the enlarged filtration $\mathcal{F}_t^\psi \triangleq \mathcal{F}_t \vee \sigma(\psi)$ or not. In the concrete examples considered in Sections 4–6 it will be shown that the filtering problem can be formulated as

$$\text{system} \qquad d\theta = 0 \qquad \theta \sim \mathcal{N}(m, v) \tag{1}$$

$$\text{observation} \qquad d\bar{B}_t = (a_\theta(t)\theta + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi)dt + d\tilde{B}_t \qquad \bar{B}_0 = 0 \tag{2}$$

$$d\psi = 0 \qquad \psi \in \{\psi_\theta, \bar{\psi}\}, \tag{3}$$

where $\tilde{B}_t$ is a Brownian motion in $\mathcal{F}_t^\psi$ and $a_\theta$, $\bar{a}$, and $a_\psi$ are deterministic functions. The information available to an investor with a signal is $\bar{\mathcal{F}}_t^\psi \triangleq \bar{\mathcal{F}}_t \vee \sigma(\psi)$. In the following a brief review of the filtering results needed for the models considered throughout this paper will be given. For a thorough treatment of filtering theory, Liptser and Shiryaev (2001a) is an excellent reference.

**Theorem 2.1** (The Kalman-Bucy filter)**.** *Let the coefficients of the system of equations in* (1)–(3) *satisfy the conditions of Liptser and Shiryaev (2001a, Subsection 10.3.1). Then*

*$\hat{\theta}_t \triangleq \mathbb{E}[\theta \,|\, \bar{\mathcal{F}}^{\psi}_t]$ and $v(t) \triangleq \mathbb{E}[(\theta - \hat{\theta}_t)^2 \,|\, \bar{\mathcal{F}}^{\psi}_t]$ are solutions to the system of equations*

$$\begin{aligned} d\hat{\theta}_t &= v(t)a_\theta(t)\left[d\bar{B}_t - \left(a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi\right)dt\right] \\ v'(t) &= -a_\theta(t)^2 v(t)^2 \end{aligned} \tag{4}$$

*with initial conditions $\hat{\theta}_0 = \mathbb{E}[\theta \,|\, \psi]$ and $v(0) = \mathbb{E}[(\theta - \hat{\theta}_0)^2 \,|\, \psi]$.*

*Proof.* Consider the slightly modified version of (1)–(3)

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observation} & d\bar{B}_t = (a_\theta(t)\theta + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi_t)dt + d\tilde{B}_t & \bar{B}_0 = 0 \\ & d\psi_t = b\,d\tilde{B}^{\psi}_t & \psi_0 \in \{\psi_\theta, \bar{\psi}\}, \end{array}$$

where $\tilde{B}^{\psi}_t$ is a Brownian motion independent of $\tilde{B}_t$ and $b$ is an arbitrary constant. This filtering problem fits into the framework of Liptser and Shiryaev (2001a, Theorem 10.3), which then states that $\hat{\theta}_t \triangleq \mathbb{E}[\theta_t \,|\, \bar{\mathcal{F}}^{\psi}_t]$ and $v(t) \triangleq \mathbb{E}[(\theta_t - \hat{\theta}_t)^2 \,|\, \bar{\mathcal{F}}^{\psi}_t]$ are the solutions to

$$\begin{aligned} d\hat{\theta}_t &= v(t)\begin{pmatrix} a_\theta(t) & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\left[\begin{pmatrix} d\bar{B}_t \\ d\psi_t \end{pmatrix} - \left(\begin{pmatrix} a_\theta(t) \\ 0 \end{pmatrix}\hat{\theta}_t + \begin{pmatrix} \bar{a}(t) & a_\psi(t) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{B}_t \\ \psi_t \end{pmatrix}\right)dt\right] \\ &= v(t)a_\theta(t)\left[d\bar{B}_t - \left(a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi_t\right)dt\right] \\ v'(t) &= -a_\theta(t)^2 v(t)^2, \end{aligned}$$

with initial conditions $\hat{\theta}_0 = \mathbb{E}[\theta \,|\, \psi_0]$ and $v(0) = \mathbb{E}[(\theta - \hat{\theta}_0)^2 \,|\, \psi_0]$. Since the solution does not depend on $b$ the result follows immediately. □

The next result proves to be essential in order to formulate the investor's optimization problem in terms of processes adapted to the filtration observable to him.

**Lemma 2.2.** *The process*

$$\hat{B}_t \triangleq \bar{B}_t - \int_0^t \left(a_\theta(s)\hat{\theta}_s + \bar{a}(s)\bar{B}_s + a_\psi(s)\psi\right)ds \tag{5}$$

*is a $(\mathbb{P}, \bar{\mathcal{F}}^{\psi}_t)$-Brownian motion.*

*Proof.* First note that from (2)

$$\hat{B}_t = \tilde{B}_t - \int_0^t a_\theta(s)(\hat{\theta}_s - \theta)ds \tag{6}$$

which then gives

$$\begin{aligned}\mathbb{E}\left[\hat{B}_t-\hat{B}_u\,\middle|\,\bar{\mathcal{F}}^{\psi}_u\right]&=\mathbb{E}\left[\tilde{B}_t-\tilde{B}_u-\int_u^t a_\theta(s)(\hat{\theta}_s-\theta)ds\,\middle|\,\bar{\mathcal{F}}^{\psi}_u\right]\\&=\mathbb{E}\left[\mathbb{E}\left[\tilde{B}_t-\tilde{B}_u\,\middle|\,\mathcal{F}^{\psi}_u\right]\,\middle|\,\bar{\mathcal{F}}^{\psi}_u\right]-\int_u^t a_\theta(s)\mathbb{E}\left[\hat{\theta}_s-\theta\,\middle|\,\bar{\mathcal{F}}^{\psi}_u\right]ds\\&=0\end{aligned}$$

since $\tilde{B}_t$ is a $\mathcal{F}^{\psi}_t$-Brownian motion and by the definiotion of $\hat{\theta}_s=\mathbb{E}\left[\theta\,|\,\bar{\mathcal{F}}^{\psi}_s\right]$. The above calculation show that $\hat{B}_t$ is a $(\mathbb{P},\bar{\mathcal{F}}^{\psi}_t)$-martingale and because the absolutely continuous part in (6) does not contribute to the quadratic variation, $\langle\hat{B},\hat{B}\rangle_t=t$, and so by the Lévy characterization of Brownian motion, $\hat{B}_t$ is a $(\mathbb{P},\bar{\mathcal{F}}^{\psi}_t)$-Brownian motion. □

**Remark.** The process, $\hat{B}_t$, defined in (5) is the so-called innovation process from filtering theory. The changes in this process, scaled by $v(t)a_\theta(t)$, captures the fluctuations/innovations in the observed market price of risk as can be seen from (4). From (5) it can be seen that the innovation process along with the signal carries the same information as the observable process, i.e. $\bar{\mathcal{F}}^{\psi}_t=\hat{\mathcal{F}}^{\psi}_t\triangleq\sigma(\hat{B}_u\,|\,0\le u\le t)\vee\sigma(\psi)$. The argument goes as follows: From (6) it follows that (2) can be written as

$$d\bar{B}_t=(a_\theta(t)\hat{\theta}_t+\bar{a}(t)\bar{B}_t+a_\psi(t)\psi)dt+d\hat{B}_t.$$

This stochastic differential equation can be solved explicitly by an application of Itô's formula to the process $\exp\left(-\int_0^t\bar{a}(s)ds\right)\bar{B}_t$. The solution is

$$\bar{B}_t=\int_0^t\exp\left(\int_s^t\bar{a}(u)du\right)\left(a_\theta(s)\hat{\theta}_s+a_\psi(s)\psi\right)ds+\hat{B}_t,$$

which shows that $(\bar{B}_t)$ is $\hat{\mathcal{F}}^{\psi}_t$-adapted, since (5) inserted in (4) gives

$$d\hat{\theta}_t=v(t)a_\theta(t)d\hat{B}_t,$$

from which it follows that $\hat{\theta}_t$ is $\hat{\mathcal{F}}_t$-adapted. Hence $\bar{\mathcal{F}}^{\psi}_t\subseteq\hat{\mathcal{F}}^{\psi}_t$. The other inclusion holds trivially from (5), since $\hat{\theta}_t=\mathbb{E}[\theta\,|\,\bar{\mathcal{F}}^{\psi}_t]$ is defined to be $\bar{\mathcal{F}}^{\psi}_t$-adapted.

# 3 Solution to a generic investor's problem

An investor will be characterized by a *utility function*, which is a strictly increasing, strictly concave, continuously differentiable mapping, $u:\mathbb{R}_+\to\mathbb{R}$, that satisfies the

Inada conditions

$$\lim_{x\to 0} u'(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} u'(x) = 0.$$

Given an initial amount of wealth, $W_0$, the investor chooses consumption and investment processes, $(c_t)$ and $(\pi_t)$, in order to maximize expected lifetime utility over the time horizon $[0, T]$. The strategy $(c_t, \pi_t)$ is called *admissible* if it ensures a lower bounded wealth process and is adapted to the information filtration, $\bar{\mathcal{F}}_t^{\psi}$. It is furthermore required that the strategy satisfies the usual integrability conditions, see e.g. Duffie (2001). The set of all admissible strategies on the interval $[t, T]$ will be denoted $\bar{\mathcal{A}}_t^{\psi}$.

Let $\iota_c, \iota_w \in \{0, 1\}$ be binary variables indicating whether the investor derives utility from intermediate consumption, terminal wealth, or both. The objective is then to maximize the expected discounted utility derived from consumption and bequest motives, i.e.

$$\sup_{(c_t,\pi_t)\in\bar{\mathcal{A}}_0^{\psi}} \mathbb{E}\left[\iota_c \int_0^T e^{-\delta t} u(c_t)dt + \iota_w e^{-\delta T} u(W_T) \,\middle|\, \psi\right]$$

subject to the wealth dynamics

$$\begin{aligned} dW_t &= W_t\left[r + \pi_t\sigma\theta\right]dt - c_t dt + W_t\pi_t\sigma dB_t \\ W_0 &= 1. \end{aligned} \tag{7}$$

Notice that the wealth dynamics is expressed in terms of $\theta$ and $B_t$ that are unobservable. For that reason, the investor cannot make any decisions based on these quantities. Recall, however, that $\bar{B}_t = \theta t + B_t$, which along with Lemma 2.2 allows (7) to be written as

$$\begin{aligned} dW_t &= W_t\left[r + \pi_t\sigma X_t\right]dt - c_t dt + W_t\pi_t\sigma d\hat{B}_t \\ W_0 &= 1, \end{aligned}$$

where $X_t \triangleq a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi$ is the market price of risk process perceived by the investor. As will be evident from the concrete examples in Sections 4–6, the dynamics for the perceived market price of risk can be represented by

$$dX_t = \hat{v}(t)d\hat{B}_t,$$

for some deterministic function $\hat{v}$ satisfying $\hat{v}'(t) = -\hat{v}(t)^2$. Since $(X_t)$ and $(\hat{B}_t)$ are observable the investor's problem fits into a complete information context and can be solved with well-developed techniques. One such method is the dynamic programming approach, which can be applied to this setting since the pair $(W_t, X_t)$ forms a Markov diffusion process, which contains all the information needed for the investor to make his consumption and investment decisions.

The indirect utility at time $t$ is therefore $J_t = J(W_t, X_t, t)$, where the function $J$ is given by

$$J(w,x,t) \triangleq \sup_{(c_s,\pi_s)\in\bar{\mathcal{A}}_t^{\psi}} \mathbb{E}_{w,x,t}\left[\iota_c \int_t^T e^{-\delta(s-t)}u(c_s)ds + \iota_w e^{-\delta(T-t)}u(W_T)\,\middle|\,\bar{\mathcal{F}}_t^{\psi}\right]$$

and satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\delta J(w,x,t) = \sup_{c\geq 0,\pi\in\mathbb{R}} \Big\{ \iota_c u(c) + \frac{\partial J}{\partial t}(w,x,t) + J_w(w,x,t)\left(w\left[r+\pi\sigma x\right] - \iota_c c\right) \\ + \frac{1}{2}J_{ww}(w,x,t)w^2\pi^2\sigma^2 + \frac{1}{2}J_{xx}(w,x,t)\hat{v}(t)^2 \\ + J_{wx}(w,x,t)w\pi\sigma\hat{v}(t)\Big\},\end{aligned} \tag{8}$$

with terminal condition $J(w,x,T) = \iota_w u(w)$. In the following it is assumed that $\iota_c = \iota_w = 1$. The first-order condition with respect to $c$ is

$$u'(c) = J_w(w,x,t),$$

which gives the candidate optimal consumption strategy as

$$c_t^* = I(J_w(W_t^*, \hat{X}_t, t)), \tag{9}$$

where $I(\cdot) \triangleq (u')^{-1}(\cdot)$ is the inverse of marginal utility, $u'(\cdot)$. The first-order condition with respect to $\pi$ is

$$J_w(w,x,t)w\sigma x + J_{ww}(w,x,t)w^2\pi\sigma^2 + J_{wx}(w,x,t)w\sigma\hat{v}(t) = 0,$$

which gives the candidate optimal investment strategy as

$$\pi_t^* = -\frac{J_w(W_t^*, \hat{X}_t, t)}{J_{ww}(W_t^*, \hat{X}_t, t)W_t^*}\frac{\hat{X}_t}{\sigma} - \frac{J_{wx}(W_t^*, \hat{X}_t, t)}{J_{ww}(W_t^*, \hat{X}_t, t)W_t^*}\frac{\hat{v}(t)}{\sigma} \tag{10}$$

Substituting the candidate strategies back into the HJB equation (8) and gathering terms yields

$$
\begin{aligned}
\delta J(w,x,t) = {} & u(I(J_w(w,x,t))) - J_w(w,x,t)I(J_w(w,x,t)) + rwJ_w(w,x,t) \\
& - \frac{1}{2}\frac{J_w(w,x,t)^2}{J_{ww}(w,x,t)}x^2 - \frac{J_w(w,x,t)J_{wx}(w,x,t)}{J_{ww}(w,x,t)}\hat{v}(t)x \\
& - \frac{1}{2}\frac{J_{wx}(w,x,t)^2}{J_{ww}(w,x,t)}\hat{v}(t)^2 + \frac{1}{2}J_{xx}(w,x,t)\hat{v}(t)^2 + \frac{\partial J}{\partial t}(w,x,t), \qquad (11)
\end{aligned}
$$

which is a highly non-linear second order partial differential equation. If this PDE has a solution $J(w,x,t)$ so that the strategy defined by (9) and (10) is admissible, then the verification theorem states that this strategy is indeed the optimal consumption and investment strategy and the function $J(w,x,t)$ is indeed the indirect utility function. With no utility from intermediate consumption, i.e. $\iota_c = 0$, the first two terms on the right-hand side of (11) vanish.

## 3.1 Power utility

Assuming that the investor derives utility according to the function

$$
u(x) = \frac{x^{1-\gamma}}{1-\gamma},
$$

the above analysis can be made more explicit. Due to the linearity of the wealth dynamics in (**??**) it seems reasonable to conjecture that if the strategy $(c_s^*, \pi_s^*)$ is optimal with time $t$ wealth $w$ and the corresponding wealth process $(W_s^*)$, then the strategy $(kc_s^*, \pi_s^*)$will be optimal with time $t$ wealth $kw$ and the corresponding wealth process $(kW_s^*)$. If this is true, then the indirect utility function is homogenous of degree $1-\gamma$ in the wealth $w$, i.e.

$$
J(kw,x,t) = k^{1-\gamma}J(w,x,t).
$$

Inserting $k = \frac{1}{w}$ and rearranging, the indirect utility function is given by

$$
J(w,x,t) = \frac{g(x,t)^{\gamma}w^{1-\gamma}}{1-\gamma}, \qquad (12)
$$

where $g(x,t)^{\gamma} = (1-\gamma)J(1,x,t)$. Finding the relevant derivatives and substituting them back into (11) and collecting terms reduces the dimensionality of the problem by one and gives the following PDE which $g$ must satisfy

$$
\begin{aligned}
0 = \iota_c & - \left(\frac{\delta}{\gamma} - \frac{1-\gamma}{\gamma}r - \frac{1-\gamma}{2\gamma^2}x^2\right)g(x,t) + \frac{\partial g}{\partial t}(x,t) \\
& + \frac{1-\gamma}{\gamma}x\hat{v}(t)\frac{\partial g}{\partial x}(x,t) + + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x,t)\hat{v}(t)^2 \qquad (13)
\end{aligned}
$$

with terminal condition $g(x,T) = \iota_w$.

**Theorem 3.1** (Utility from terminal wealth only)**.** *For an investor with power utility from terminal wealth only, the indirect utility function is given by*

$$J(w,x,t) = e^{-\delta(T-t)}\frac{1}{1-\gamma}\left(we^{A_1(t;T)+\frac{1}{2}A_3(t;T)x^2}\right)^{1-\gamma}, \tag{14}$$

*where $A_3$ solves the ordinary differential equation*

$$0 = \frac{1}{2\gamma} + \frac{1-\gamma}{\gamma}\hat{v}(t)A_3(t;T) + \frac{1}{2}A_3'(t;T) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2A_3(t;T)^2 \tag{15}$$

*with the boundary condition $A_3(T;T) = 0$ and $A_1$ is given by*

$$A_1(t;T) = r(T-t) + \frac{1}{2}\int_t^T \hat{v}(s)^2A_3(s;T)ds. \tag{16}$$

*The optimal investment strategy is given by*

$$\Pi(w,x,t) = \frac{1}{\gamma}\frac{x}{\sigma} + \frac{1-\gamma}{\gamma}\frac{\hat{v}(t)A_3(t;T)x}{\sigma}. \tag{17}$$

*Proof.* Setting $\iota_c = 0$ and $\iota_w = 1$ in (13), a candidate solution of the form $g(x,t) = e^{h(x,t;T)}$ with

$$h(x,t;T) = -\frac{\delta}{\gamma}(T-t) + \frac{1-\gamma}{\gamma}A_1(t;T) + \frac{1-\gamma}{\gamma}A_2(t;T)x + \frac{1-\gamma}{2\gamma}A_3(t;T)x^2$$

is considered. Since $g(x,T) = 1$ the auxiliary functions necessarily must satisfy $A_1(T;T) = A_2(T;T) = A_3(T;T) = 0$. Taking the relevant derivatives

$$\begin{aligned}
\frac{\partial g}{\partial t}(x,t) &= g(x,t)\frac{1-\gamma}{\gamma}\left(\frac{\delta}{1-\gamma} + A_1'(t;T) + A_2'(t;T)x + \frac{1}{2}A_3'(t;T)x^2\right)\\
\frac{\partial g}{\partial x}(x,t) &= g(x,t)\frac{1-\gamma}{\gamma}\left[A_2(t;T) + A_3(t;T)x\right]\\
\frac{\partial^2 g}{\partial x^2}(x,t) &= g(x,t)\left(\frac{1-\gamma}{\gamma}\right)^2\left[A_2(t;T) + A_3(t;T)x\right]^2 + g(x,t)\frac{1-\gamma}{\gamma}A_3(t;T)
\end{aligned}$$

and substituting them back into (13) gives

$$\begin{aligned}
0 = r &+ \frac{1}{2\gamma}x^2 + \frac{1-\gamma}{\gamma}\hat{v}(t)x\left[A_2(t;T) + A_3(t;T)x\right]\\
&+ A_1'(t;T) + A_2'(t;T)x + \frac{1}{2}A_3'(t;T)x^2\\
&+ \frac{1}{2}\hat{v}(t)^2A_3(t;T) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2\left[A_2(t;T) + A_3(t;T)x\right]^2.
\end{aligned} \tag{18}$$

Collecting terms involving $x^2$ and $x$ leads to

$$\begin{aligned}0 = &\left(r + A_1'(t;T) + \frac{1}{2}\hat{v}(t)^2 A_3(t;T) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2 A_2(t;T)^2\right) \\ &+ \left(\frac{1-\gamma}{\gamma}\hat{v}(t) A_2(t;T) + A_2'(t;T) + \frac{1-\gamma}{\gamma}\hat{v}(t)^2 A_2(t;T) A_3(t;T)\right) x \\ &+ \left(\frac{1}{2\gamma} + \frac{1-\gamma}{\gamma}\hat{v}(t) A_3(t;T) + \frac{1}{2}A_3'(t;T) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2 A_3(t;T)^2\right) x^2,\end{aligned}$$

which must hold for all $(x,t)$. Therefore all the big parentheses must equal zero leaving three ordinary differential equations in three unknown functions to be solved. The second ODE readily gives $A_2(t;T) = 0$, which substituted into the first and third ODE gives (15)–(16). (14) is then easily seen from (12) and finding the relevant derivatives of the indirect utility function and inserting them into (10) produces (17). □

**Theorem 3.2** (Utility from intermediate consumption and possibly terminal wealth)**.** *For an investor with power utility from intermediate consumption and possibly terminal wealth, the indirect utility function is given by*

$$J(w,x,t) = \frac{1}{1-\gamma}\left(\int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)}\right)^{\gamma} w^{1-\gamma},$$

*where*

$$h(x,t;T) = -\frac{\delta}{\gamma}(T-t) + \frac{1-\gamma}{\gamma}\left(A_1(t;T) + \frac{1}{2}A_3(t;T)x^2\right),$$

$A_3$ *solves the ordinary differential equation* (15) *with* $A_3(T;T) = 0$*, and* $A_1$ *is given by* (16)*. The optimal investment strategy is given by*

$$\Pi(w,x,t) = \frac{1}{\gamma}\frac{x}{\sigma} + \frac{1-\gamma}{\gamma}\frac{\hat{v}(t)x}{\sigma}\frac{\int_t^T e^{h(x,t;s)} A_3(t;s) ds + \iota_w e^{h(x,t;T)} A_3(t;T)}{\int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)}}$$

*and the optimal consumption strategy is given by*

$$C(w,x,t) = \left(\int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)}\right)^{-1} w.$$

*Proof.* As for the case with utility from terminal wealth only, a qualified guess is

$$g(x,t) = \int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)}.$$

Finding the relevant derivatives

$$
\begin{aligned}
\frac{\partial g}{\partial t}(x,t) &= -1 \\
&\quad + \frac{1-\gamma}{\gamma}\int_t^T \left(\frac{\delta}{1-\gamma} + A_1'(t;s) + A_2'(t;s)x + \frac{1}{2}A_3'(t;s)x^2\right) e^{h(x,t;s)}ds \\
&\quad + \iota_w \frac{1-\gamma}{\gamma}\left(\frac{\delta}{1-\gamma} + A_1'(t;T) + A_2'(t;T)x + \frac{1}{2}A_3'(t;T)x^2\right) e^{h(x,t;T)} \\
\frac{\partial g}{\partial x}(x,t) &= \frac{1-\gamma}{\gamma}\int_t^T \left[A_2(t;s) + A_3(t;s)x\right] e^{h(x,t;s)}ds \\
&\quad + \frac{1-\gamma}{\gamma}\iota_w [A_2(t;T) + A_3(t;T)x] e^{h(x,T;T)} \\
\frac{\partial^2 g}{\partial x^2}(x,t) &= \left(\frac{1-\gamma}{\gamma}\right)^2 \int_t^T \left[A_2(t;s) + A_3(t;s)x\right]^2 e^{h(x,t;s)}ds \\
&\quad + \left(\frac{1-\gamma}{\gamma}\right)^2 \iota_w [A_2(t;T) + A_3(t;T)x]^2 e^{h(x,T;T)} \\
&\quad + \frac{1-\gamma}{\gamma}\int_t^T A_3(t;s)e^{h(x,t;s)}ds + \frac{1-\gamma}{\gamma}\iota_w A_3(t;T)e^{h(x,T;T)}
\end{aligned}
$$

and substituting them back into (13) gives

$$
\begin{aligned}
0 = \int_t^T \Bigg( & r + \frac{1}{2\gamma}x^2 + \frac{1-\gamma}{\gamma}\hat{v}(t)x\left[A_2(t;s) + A_3(t;s)x\right] \\
& A_1'(t;s) + A_2'(t;s)x + \frac{1}{2}A_3'(t;s)x^2 \\
& \frac{1}{2}\hat{v}(t)^2 A_3(t;s) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2\left[A_2(t;s) + A_3(t;s)x\right]^2 \Bigg) e^{h(x,t;s)}ds \\
+ \iota_w \Bigg( & r + \frac{1}{2\gamma}x^2 + \frac{1-\gamma}{\gamma}\hat{v}(t)x\left[A_2(t;T) + A_3(t;T)x\right] \\
& + A_1'(t;T) + A_2'(t;T)x + \frac{1}{2}A_3'(t;T)x^2 \\
& + \frac{1}{2}\hat{v}(t)^2 A_3(t;T) + \frac{1-\gamma}{2\gamma}\hat{v}(t)^2\left[A_2(t;T) + A_3(t;T)x\right]^2 \Bigg).
\end{aligned}
$$

The terms in the big parentheses are identical to (18) for which reason the auxiliary functions from the no consumption case also solves this system. □

Theorems 3.1–3.2 provide the solution of a generic investor. In Sections 4–6, the theorems will be applied to investors receiving either no signal, a signal about the expected stock return (through $\theta$), or a signal about the terminal stock price (through $\bar{B}_T$) respectively.

# 4 Solution to the uninformed investor's problem

In order to apply Theorems 3.1–3.2 the uninformed investor's filtering problem should be on the form (1)–(3). Without a signal, however, the filtering problem is given as

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observation} & d\bar{B}_t = \theta dt + dB_t & \bar{B}_0 = 0, \end{array}$$

which fits into (1)–(3) with $a_\theta(t) = 1$, $\bar{a}(t) = a_\psi(t) = 0$ and $\tilde{B}_t = B_t$. With this specification the solution to the filtering problem is readily given by Theorem 2.1.

**Corollary 4.1.** *The solution of the filtering problem satisfies the system of equations*

$$\begin{aligned} d\hat{\theta}_t &= v(t)\left[d\bar{B}_t - \hat{\theta}_t dt\right] & \hat{\theta}_0 &= m \\ v'(t) &= -v(t)^2 & v(0) &= v, \end{aligned} \tag{19}$$

*which yield the solution*

$$\begin{aligned} \hat{\theta}_t &= v(t)\left[\frac{m}{v} + \bar{B}_t\right] \\ v(t) &= \frac{v}{1+vt}. \end{aligned}$$

*Proof.* Apply Theorem 2.1 with $a_\theta(t) = 1$, $\bar{a}(t) = a_\psi(t) = 0$ and $\bar{\mathcal{F}}_t^\psi = \bar{\mathcal{F}}_t$. The explicit solution to the ordinary differential equation is easily verified. The stochastic differential equation can be solved by an application of Itô's lemma to the process $\exp\left(\int_0^t v(s)ds\right)\hat{\theta}_t$, which gives

$$\begin{aligned} d\left(\exp\left(\int_0^t v(s)ds\right)\hat{\theta}_t\right) &= \exp\left(\int_0^t v(s)ds\right)\left[v(t)\hat{\theta}_t dt + d\hat{\theta}_t\right] \\ &= \exp\left(\int_0^t v(s)ds\right)v(t)d\bar{B}_t. \end{aligned} \tag{20}$$

Realizing that

$$\exp\left(\int_0^t v(s)ds\right) = \exp\left(\log(1+vt)\right) = \frac{v}{v(t)},$$

the result can be obtained by writing (20) on integral form

$$\frac{v}{v(t)}\hat{\theta}_t = \hat{\theta}_0 + v\bar{B}_t = m + v\bar{B}_t$$

and rearranging appropriately. □

By Lemma 2.2 the innovation process $\hat{B}_t \triangleq \bar{B}_t - \int_0^t \hat{\theta}_s ds$ is a $(\mathbb{P}, \bar{\mathcal{F}}_t)$-Brownian motion.

Noticing further that the market price of risk process perceived by the investor $X_t = a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi = \hat{\theta}_t$ has dynamics given by (19)

$$dX_t = v(t)\left[d\bar{B}_t - \hat{\theta}_t dt\right] = v(t)d\hat{B}_t,$$

the results in Section 3 can be applied with $\hat{v} = v$.

**Proposition 4.2** (Utility from terminal wealth only). *For an investor with power utility from terminal wealth only, the indirect utility function is given by*

$$J_0(w,x,t;T) = \exp\left(-\left[\delta - r(1-\gamma)\right](T-t) + \frac{1-\gamma}{2\gamma}\frac{T-t}{1+(1-\frac{1}{\gamma})v(t)(T-t)}x^2\right)$$
$$\times\left(\frac{\left[1+v(t)(T-t)\right]^{1-\frac{1}{\gamma}}}{1+(1-\frac{1}{\gamma})v(t)(T-t)}\right)^{\frac{\gamma}{2}}\frac{w^{1-\gamma}}{1-\gamma}$$

*and the optimal investment strategy is given by*

$$\begin{aligned}\Pi_0(w,x,t;T) &= \frac{1}{\gamma}\frac{x}{\sigma}\left[1 - \frac{(1-\frac{1}{\gamma})v(t)(T-t)}{1+(1-\frac{1}{\gamma})v(t)(T-t)}\right]\\ &= \frac{1}{\gamma}\frac{x}{\sigma}\frac{1}{1+(1-\frac{1}{\gamma})v(t)(T-t)}.\end{aligned}$$

*Proof.* Applying Theorem 3.1 with $\hat{v}(t) = v(t)$, the solutions to the system of ordinary differential equations (15)–(16) are

$$A_3(t;T) = \frac{1}{\gamma}\frac{T-t}{1+(1-\frac{1}{\gamma})v(t)(T-t)} \tag{21}$$

$$A_1(t;T) = r(T-t) + \frac{\gamma}{1-\gamma}\log\left(\sqrt{\frac{\left[1+v(t)(T-t)\right]^{1-\frac{1}{\gamma}}}{1+(1-\frac{1}{\gamma})v(t)(T-t)}}\right). \tag{22}$$

which inserted into (14) and (17) gives the result. □

**Remark.** For the solution to be well-defined it is required that the relative risk aversion coefficient of the investor satisfies

$$1 + \left(1 - \frac{1}{\gamma}\right)v(t)(T-t) > 0,$$

which simplifies to

$$\gamma > \frac{v(t)(T-t)}{1+v(t)(T-t)}$$

for all $t$. It can be shown that the right-hand side in this expression is decreasing in $t$, and hence it is sufficient to check the restriction for $t = 0$.

**Remark.** When the investor knows the expected return with certainty, corresponding to $v(t) = 0$, investment opportunities are constant and the solution simplifies to the well-known result of Merton (1969). With parameter uncertainty $v(t) > 0$, and the investor will hedge against unfavorable movements in the perceived market price of risk when $\gamma > 1$ and speculate for favorable shifts in the perceived market price of risk when $\gamma < 1$. This well-documented effect (Samuelson, 1991) is increasing in the level of uncertainty, $v(t)$, and the investment horizon, $T$. This seems reasonable since both factors imply more uncertainty, which a non-myopic investor will take into account when the investment decision is to be made. Depending on the level of parameter uncertainty, the indirect utility function can be higher or lower than the indirect utility obtainable with full information. The intuition is that the investor effectively perceives the expected rate of return on the stock as a Gaussian process, and therefore assigns positive probability to very high (low) stock returns, which improves (aggravates) the indirect utility obtainable. In the following example we will study the above phenomena in more detail in order to obtain a better understanding of parameter uncertainty.

**Example 4.3.** The benchmark parameters for this and the example in Section 5 are as follows: The financial market is characterized by a risk-free rate of return $r = 0.04$ and the volatility of the stock return is $\sigma = 0.2$. The investor has a time-preference rate of $\delta = 0.02$, initial wealth $W_0 = 1$, and a Gaussian prior distribution on $\theta \sim \mathcal{N}(m, v)$ with $m = 0.20$ and $v = 0.04$ corresponding to a prior distribution on the expected rate of return, $\mu$, with mean 8% and standard deviation 4%.

Figure 1 illustrates the initial wealth necessary in an economy with parameter uncertainty to obtain the same indirect utility as in an economy with complete information. A level above (below) one indicates a loss (gain) in terms of indirect utility, which indicates that the investor would prefer to be fully (incompletely) informed. Utility losses (gains) are observed for investors with relative risk aversion coefficients above (below) two in the scenarios implemented, but one could find the value $\gamma^*$, that gives the same indirect utility, numerically. For high coefficients of risk aversion the effect is dampened by the fact, that the portfolio allocation is shifted towards the risk free investment and is therefore less exposed to parameter uncertainty. For $\gamma > \gamma^*$ parameter uncertainty is disliked by the investor and the effect is increasing in the coefficient of risk aversion and the time horizon.

The optimal stock holdings for the case with and without parameter uncertainty are depicted in Figure 2. In the full information economy the optimal policy is to invest a constant fraction of wealth in each asset. With parameter uncertainty, however, the optimal stock holding is decreasing (increasing) in the investment horizon for investor with a relative risk aversion coefficient greater (less) than one, as was also noted in the

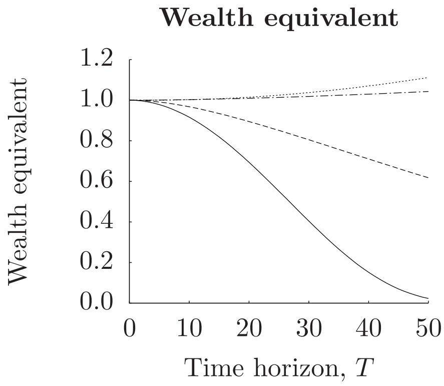


Figure 1: Initial wealth needed in an economy with parameter uncertainty to obtain the indirect utility in a full information economy. The solid line is for $\gamma = 0.75$, the dashed line is for $\gamma = 0.99$, the dotted line is for $\gamma = 2$, and the dashed-dotted line is for $\gamma = 10$.

remark above. The difference between the two depicted strategies is the hedge/speculative term, which is increasing in the time horizon.

**Proposition 4.4** (Utility from intermediate consumption and possibly terminal wealth). *For an investor with power utility from intermediate consumption and possibly terminal wealth, the indirect utility function is given by*

$$J(w, x, t; T) = \frac{1}{1-\gamma} \left( \int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)} \right)^\gamma w^{1-\gamma},$$

*where*

$$h(x, t; T) = -\frac{\delta}{\gamma}(T - t) + \frac{1-\gamma}{\gamma} \left( A_1(t; T) + \frac{1}{2} A_3(t; T) x^2 \right),$$

$A_1$ *is given by* (22) *and* $A_3$ *is given by* (21). *The optimal investment strategy is given by*

$$\Pi(w, x, t; T) = \frac{1}{\gamma}\frac{x}{\sigma} + \frac{1-\gamma}{\gamma}\frac{v(t)x}{\sigma} \frac{\int_t^T e^{h(x,t;s)} A_3(t; s) ds + \iota_w e^{h(x,t;T)} A_3(t; T)}{\int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)}}$$

*and the optimal consumption strategy is given by*

$$C(w, x, t; T) = \left( \int_t^T e^{h(x,t;s)} ds + \iota_w e^{h(x,t;T)} \right)^{-1} w.$$

*Proof.* Apply Theorem 3.2 with $\hat{v}(t) = v(t)$. □

**Remark.** Rearranging the equations in the above proposition allows the optimal invest-

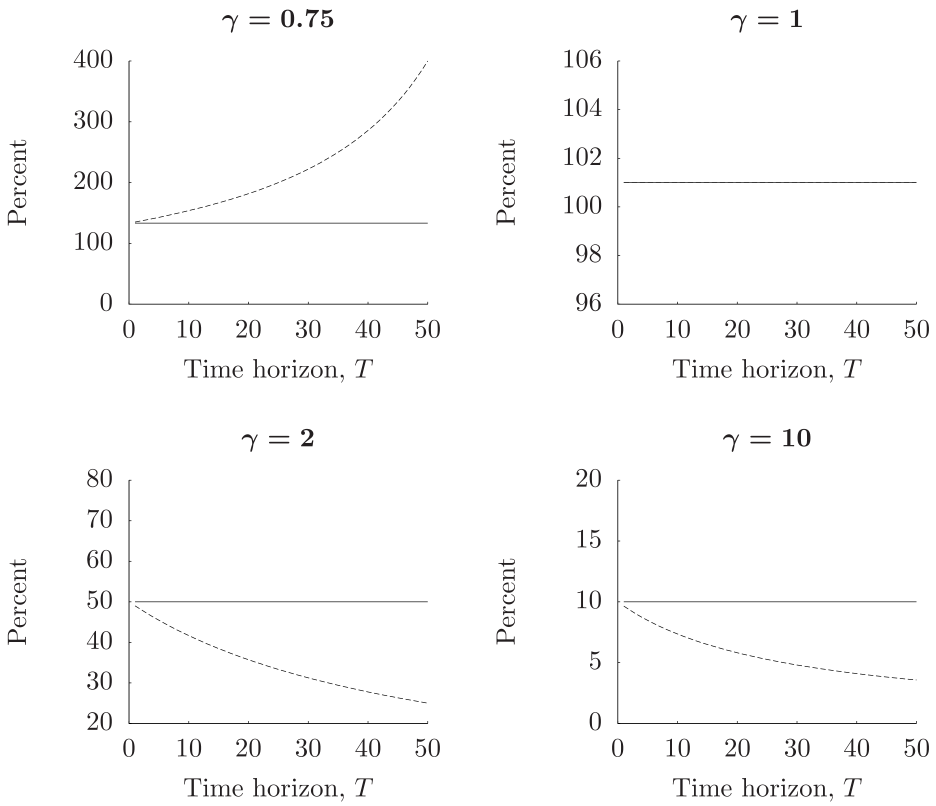


Figure 2: Fraction of wealth invested in the stock. The solid line is for the full information benchmark and the dashed line is for the incomplete information case.

ment strategy to be expressed as

$$\Pi(w,x,t;T) = \frac{\int_t^T e^{h(x,t;s)}\Pi_0(w,x,t;s)ds + \iota_w e^{h(x,t;T)}\Pi_0(w,x,t;T)}{\int_t^T e^{h(x,t;s)}ds + \iota_w e^{h(x,t;T)}}.$$

When $\iota_w = 0$ the optimal investment strategy simplifies to

$$\Pi(w,x,t;T) = \int_t^T w(x,t;s)\Pi_0(w,x,t;s)ds$$

where $w(x,t;s) \triangleq \frac{e^{h(x,t;s)}}{\int_t^T e^{h(x,t;s)}ds}$. As in Wachter (2002), this allows the optimal investment strategy of an investor with utility from intermediate consumption and an investment horizon of $T$ to be interpreted as a weighted average of the optimal investment strategies of investors with investment horizons $s \in [t,T]$ and utility from terminal wealth only. Therefore the effect of parameter uncertainty will be the same as with utility from terminal wealth only. Since the optimal investment strategy for an investor with utility from terminal wealth only is decreasing (increasing) for $\gamma > 1$ ($\gamma < 1$), an investor with utility from intermediate consumption will invest more (less) than the corresponding investor with utility from terminal wealth for a given time horizon.

**Remark.** Unfortunately, the solution with intermediate consumption is too complicated to be computed in closed form. In the examples to come the integrals have therefore been computed numerically according to Simpson's rule.

# 5 Signal about expected return

Having considered the uninformed's optimization problem in the previous section, the effects of introducing a signal about the expected rate of return are now studied. Consider therefore a signal on the form

$$\psi_\theta = \theta + \varepsilon_\theta,$$

where $\varepsilon_\theta \sim \mathcal{N}(0, \sigma_{\varepsilon_\theta}^2)$ is independent of $\mathcal{F}_t^B$ and $\theta$. Since the signal is correlated with the unobservable market price of risk, it provides the investor with more information, based on which a more qualified estimation of the market price of risk can be carried out. The filtering problem can be stated as

| | | |
|---|---|---|
| system | $d\theta = 0$ | $\theta \sim \mathcal{N}(m,v)$ |
| observations | $d\bar{B}_t = \theta dt + dB_t$ | $\bar{B}_0 = 0$ |
| | $d\psi = 0$ | $\psi_0 = \psi_\theta = \theta + \varepsilon_\theta,$ |

which fits into (1)–(3) with $a_\theta(t) = 1$, $\bar{a}(t) = a_\psi(t) = 0$ and $\tilde{B}_t = B_t$. At this point it should be noted that $B_t$ is still a Brownian motion with respect to the filtration $\mathcal{F}_t^\psi$ since $\varepsilon$ is independent of $\mathcal{F}_t^B$. As an immediate consequence of Theorem 2.1, the following Corollary is easily proven.

**Corollary 5.1.** *The solution of the filtering problem satisfies the system of equations*

$$\begin{aligned} d\hat{\theta}_t &= v_\theta(t)\left[d\bar{B}_t - \hat{\theta}_t dt\right] & \hat{\theta}_0 &= \mathbb{E}[\theta \mid \psi_\theta] \\ v'_\theta(t) &= -v_\theta(t)^2 & v_\theta(0) &= \mathbb{E}[(\theta - \hat{\theta}_0)^2 \mid \psi_\theta], \end{aligned} \tag{23}$$

*which yield the solution*

$$\begin{aligned} \hat{\theta}_t &= v_\theta(t)\left[\frac{\hat{\theta}_0}{v_\theta(0)} + \bar{B}_t\right] \\ v_\theta(t) &= \frac{v_\theta(0)}{1 + v_\theta(0)t}, \end{aligned} \tag{24}$$

*with initial conditions* $\hat{\theta}_0 = \frac{m\sigma_{\varepsilon_\theta}^2 + v\psi_\theta}{v + \sigma_{\varepsilon_\theta}^2}$ *and* $v_\theta(0) = \frac{v\sigma_{\varepsilon_\theta}^2}{v + \sigma_{\varepsilon_\theta}^2}$.

*Proof.* Except from the stated conditional prior, the proof resembles the proof of Corollary 4.1 and is omitted. To show that the stated conditional prior is correct, apply Liptser and Shiryaev (2001b, Theorem 13.1) to get

$$\begin{aligned} \hat{\theta}_0 &= \mathbb{E}[\theta] + \frac{\operatorname{cov}(\theta, \psi_\theta)}{\mathbb{V}[\psi_\theta]}(\psi_\theta - \mathbb{E}[\psi_\theta]) = m + \frac{v}{v + \sigma_{\varepsilon_\theta}^2}(\psi_\theta - m) \\ v_\theta(0) &= \mathbb{V}[\theta] - \frac{\operatorname{cov}(\theta, \psi_\theta)^2}{\mathbb{V}[\psi_\theta]} = v - \frac{v^2}{v + \sigma_{\varepsilon_\theta}^2} = \frac{v\sigma_{\varepsilon_\theta}^2}{v + \sigma_{\varepsilon_\theta}^2}, \end{aligned}$$

from which the claim follows. □

**Remark.** The conditional prior can also be expressed in terms of the correlation between the market price of risk and the signal since

$$\rho \triangleq \operatorname{corr}(\theta, \psi_\theta) = \frac{\operatorname{cov}(\theta, \psi_\theta)}{\sqrt{\mathbb{V}[\theta]\mathbb{V}[\psi_\theta]}} = \sqrt{\frac{\operatorname{cov}(\theta, \psi_\theta)^2}{\mathbb{V}[\theta]\mathbb{V}[\psi_\theta]}} = \sqrt{\frac{v}{v + \sigma_{\varepsilon_\theta}^2}}$$

implies that $\hat{\theta}_0 = (1 - \rho^2)m + \rho^2\psi_\theta$ and $v_\theta(0) = (1 - \rho^2)v$. This allows the conditional prior to be interpreted in terms of the correlation between the unobservable market price of risk and the signal. The more these are correlated (positive or negative), the better the signal in the sense that the conditional prior has a mean close to the value of the signal and a variance close to zero. On the other hand, if the unobservable market price of risk and the signal are uncorrelated, the signal does not provide any new information to the investor.

**Remark.** Equation (24) supports the intuitive understanding that a time zero signal about the market price of risk only change the estimate through an updated prior distribution on $\theta$.

By Lemma 2.2 the innovation process $\hat{B}_t \triangleq \bar{B}_t - \int_0^t \hat{\theta}_s ds$ is a $(\mathbb{P}, \bar{\mathcal{F}}_t^{\psi})$-Brownian motion. Noticing further that the market price of risk process perceived by the investor $X_t = a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi = \hat{\theta}_t$ has dynamics given by (23)

$$dX_t = v_\theta(t) \left[ d\bar{B}_t - \hat{\theta}_t dt \right] = v_\theta(t) d\hat{B}_t,$$

the results in Section 3 can be applied with $\hat{v} = v_\theta$.

**Remark.** Notice that the only difference between an uninformed and an informed investor is the prior distribution on which they base their estimations. Where the uninformed investor uses the unconditional distribution of $\theta$, the informed investor uses the conditional distribution of $\theta$ given the signal $\psi$. Therefore, all the propositions from Section 4 can be restated for an informed investor just by replacing the unconditional prior distribution with the conditional prior distribution.

**Proposition 5.2** (Utility from terminal wealth only). *For an investor with power utility from terminal wealth only, the indirect utility function after receiving the signal $\psi_\theta$ is given by*

$$J(w, x, t \,|\, \psi_\theta) = \exp\left( -\left[\delta - r(1-\gamma)\right](T-t) + \frac{1-\gamma}{2\gamma} \frac{T-t}{1 + (1-\frac{1}{\gamma}) v_\theta(t)(T-t)} x^2 \right)$$
$$\times \left( \frac{\left[1 + v_\theta(t)(T-t)\right]^{1-\frac{1}{\gamma}}}{1 + (1-\frac{1}{\gamma}) v_\theta(t)(T-t)} \right)^{\frac{\gamma}{2}} \frac{w^{1-\gamma}}{1-\gamma}.$$

*The optimal investment strategy is given by*

$$\begin{aligned}\Pi(w, x, t) &= \frac{1}{\gamma}\frac{x}{\sigma} \left[ 1 - \frac{(1-\frac{1}{\gamma}) v_\theta(t)(T-t)}{1 + (1-\frac{1}{\gamma}) v_\theta(t)(T-t)} \right] \\ &= \frac{1}{\gamma}\frac{x}{\sigma} \frac{1}{1 + (1-\frac{1}{\gamma}) v_\theta(t)(T-t)}.\end{aligned}$$

*Proof.* Resembles the proof of Proposition 4.2 and is omitted. □

**Remark.** Depending on the value of the signal, the indirect utility derived given the signal can either be above or below the indirect utility without the signal. With the signal, however, the stock holding will always be higher (lower) than without the signal for relative risk aversion coefficients higher (lower) than one, since the conditional prior variance is lower.

To be able to compute the value to an investor receiving the signal $\psi_\theta$ about the market price of risk, the ex-ante expected indirect utility is defined as

$$J_\theta(w) = \int_{\mathbb{R}} J(w, \hat{\theta}_0, 0 \,|\, \psi) p_\theta(\psi) d\psi,$$

where the probability density function of $\psi_\theta$, $p_\theta$, is given by

$$p_\theta(\psi) \triangleq \frac{1}{\sqrt{2\pi(v + \sigma_{\varepsilon_\theta}^2)}} \exp\left(-\frac{(\psi - m)^2}{2(v + \sigma_{\varepsilon_\theta}^2)}\right).$$

**Proposition 5.3.** *The ex-ante expected indirect utility function of an investor receiving the signal $\psi_\theta$ is given by*

$$J_\theta(w) = \frac{w^{1-\gamma}}{1-\gamma}\left(\frac{[1 + v_\theta(0)T]^{1-\frac{1}{\gamma}}}{1 + (1-\frac{1}{\gamma})v_\theta(0)T}\right)^{\frac{\gamma}{2}} e^{-[\delta - r(1-\gamma)]T}$$
$$\times \exp\left(\frac{1-\gamma}{2\gamma}\frac{T}{1 + (1-\frac{1}{\gamma})vT}m^2\right)\sqrt{\frac{1 + (1-\frac{1}{\gamma})v_\theta(0)T}{1 + (1-\frac{1}{\gamma})vT}}.$$

*Proof.* From Proposition 5.2

$$\int_{\mathbb{R}} J(w, \hat{\theta}_0, 0 \,|\, \psi) p_\theta(\psi) d\psi = \frac{w^{1-\gamma}}{1-\gamma}\left(\frac{[1 + v_\theta(0)(T-t)]^{1-\frac{1}{\gamma}}}{1 + (1-\frac{1}{\gamma})v_\theta(0)T}\right)^{\frac{\gamma}{2}} e^{-[\delta - r(1-\gamma)]T}$$
$$\times \int_{\mathbb{R}} \exp\left(\frac{1-\gamma}{2\gamma}\frac{T}{1 + (1-\frac{1}{\gamma})v_\theta(0)T}\hat{\theta}_0^2\right) p_\theta(\psi) d\psi, \qquad (25)$$

where $\hat{\theta}_0 = \rho^2\psi + (1-\rho^2)m$ and $v_\theta(0) = (1-\rho^2)v$ by the Remark following Corollary 5.1. The integrand on the right hand side can be expressed as

$$\frac{1}{\sqrt{2\pi(v + \sigma_{\varepsilon_\theta}^2)}} \exp\left(-\frac{1}{2(v + \sigma_{\varepsilon_\theta}^2)}\left[(\psi - m)^2 - C\hat{\theta}_0^2\right]\right),$$

where the constant $C$ is defined as

$$C \triangleq \frac{1-\gamma}{\gamma}\frac{(v + \sigma_{\varepsilon_\theta}^2)T}{1 + (1-\frac{1}{\gamma})v_\theta(0)T}.$$

The term in square brackets can be rearranged to obtain

$$(\psi - m)^2 - C\hat{\theta}_0^2 = \left[\sqrt{1 - C\rho^4}\psi - \frac{1 + C\rho^2(1-\rho^2)}{\sqrt{1 - C\rho^4}}m\right]^2 - \frac{C}{1 - C\rho^4}m^2.$$

The integral can now be computed as

$$
\begin{aligned}
&\int_{\mathbb{R}} \exp\left(\frac{1-\gamma}{2\gamma}\frac{T}{1+(1-\frac{1}{\gamma})v_\theta(0)T}\hat{\theta}_0^2\right)p_\theta(\psi)d\psi \\
&= \exp\left(\frac{1}{2(v+\sigma_{\varepsilon_\theta}^2)}\frac{C}{1-C\rho^4}m^2\right) \\
&\quad\times \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(v+\sigma_{\varepsilon_\theta}^2)}}\exp\left(-\frac{1-C\rho^4}{2(v+\sigma_{\varepsilon_\theta}^2)}\left[\psi-\frac{1+C\rho^2(1-\rho^2)}{1-C\rho^4}m\right]^2\right)d\psi \\
&= \exp\left(\frac{1}{2(v+\sigma_{\varepsilon_\theta}^2)}\frac{C}{1-C\rho^4}m^2\right)\frac{1}{\sqrt{1-C\rho^4}}, \qquad (26)
\end{aligned}
$$

since the integrand can be recognized as $\frac{1}{\sqrt{1-C\rho^4}}$ multiplied by the probability density function of a $\mathcal{N}\left(\frac{1+C\rho^2(1-\rho^2)}{1-C\rho^4}m, \frac{v+\sigma_{\varepsilon_\theta}^2}{1-C\rho^4}\right)$ random variable. Noticing that

$$
\begin{aligned}
1-C\rho^4 &= 1-\frac{1-\gamma}{\gamma}\frac{(v+\sigma_{\varepsilon_\theta}^2)T}{1+(1-\frac{1}{\gamma})v_\theta(0)T}\left(\frac{v}{v+\sigma_{\varepsilon_\theta}^2}\right)^2 \\
&= 1+\frac{(1-\frac{1}{\gamma})\rho^2 vT}{1+(1-\frac{1}{\gamma})(1-\rho^2)vT} \\
&= \frac{1+(1-\frac{1}{\gamma})vT}{1+(1-\frac{1}{\gamma})(1-\rho^2)vT},
\end{aligned}
$$

equation (26) simplifies to

$$
\exp\left(\frac{1-\gamma}{2\gamma}\frac{T}{1+(1-\frac{1}{\gamma})vT}m^2\right)\sqrt{\frac{1+(1-\frac{1}{\gamma})(1-\rho^2)vT}{1+(1-\frac{1}{\gamma})vT}}. \qquad (27)
$$

The desired result can now be obtained by combining equations (25) and (27). □

**Proposition 5.4** (Utility from intermediate consumption and possibly terminal wealth)**.** *For an investor with power utility from intermediate consumption and possibly terminal wealth, the indirect utility function after receiving the signal* $\psi_\theta$ *is given by*

$$
J(w,x,t\,|\,\psi_\theta) = \frac{1}{1-\gamma}\left(\int_t^T e^{h_\theta(x,t;s)}ds + \iota_w e^{h_\theta(x,t;T)}\right)^\gamma w^{1-\gamma},
$$

*where*

$$
h_\theta(x,t;T) = -\frac{\delta}{\gamma}(T-t)+\frac{1-\gamma}{\gamma}\left(A_{1,\theta}(t;T)+\frac{1}{2}A_{3,\theta}(t;T)x^2\right),
$$

$A_{1,\theta}$ *is given by* (22) *and* $A_{3,\theta}$ *is given by* (21) *with* $v(t)$ *replaced by* $v_\theta(t)$*. The optimal* *investment strategy is given by*

$$\Pi(w,x,t\,|\,\psi_\theta) = \frac{1}{\gamma}\frac{x}{\sigma} + \frac{1-\gamma}{\gamma}\frac{v_\theta(t)x}{\sigma}\frac{\int_t^T e^{h_\theta(x,t;s)}A_{3,\theta}(t;s)ds + \iota_w e^{h_\theta(x,t;T)}A_{3,\theta}(t;T)}{\int_t^T e^{h_\theta(x,t;s)}ds + \iota_w e^{h_\theta(x,t;T)}}$$

*and the optimal consumption strategy is given by*

$$C(w,x,t\,|\,\psi_\theta) = \left(\int_t^T e^{h_\theta(x,t;s)}ds + \iota_w e^{h_\theta(x,t;T)}\right)^{-1} w.$$

*Proof.* Resembles the proof of Proposition 4.4 and is omitted. □

**Remark.** As noted earlier, the integrals involved in the solution of the optimization problem with intermediate consumption, are too complicated to solve in closed form. The computation of the ex-ante expected utility adds to the complexity, since it involves another integration to obtain the expected indirect utility, namely

$$J_\theta(w) = \frac{w^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}}\left(\int_0^T e^{h_\theta(\hat{\theta}_0,0;s)}ds + \iota_w e^{h_\theta(\hat{\theta}_0,0;T)}\right)^\gamma p_\theta(\psi)d\psi.$$

In our examples the outer integral is computed numerically in accordance with the trapezoid rule.

## 5.1 The value of private information

The question of how much an investor is willing to pay to receive a private signal about the expected return of the stock is now addressed.

**Definition 5.5.** The value of the signal $\psi$ is defined as the fraction of wealth, $z$, an uninformed investor is willing to pay to become informed. Letting $J_0$ and $J_\psi$ denote the expected utility for an uninformed and informed investor respectively, $z$ must satisfy

$$J_0(w) = J_\psi(w(1-z)) = J_\psi(w)(1-z)^{1-\gamma},$$

i.e.

$$z = 1 - \left(\frac{J_0(w)}{J_\psi(w)}\right)^{\frac{1}{1-\gamma}}. \tag{28}$$

**Remark.** The amount an uninformed investor is willing to pay for the signal is given by $zw$. This amount is usually referred to as a certainty equivalent since giving up this (certain) amount will make the investor indifferent between having the signal and not.

**Proposition 5.6** (Utility from terminal wealth only)**.** *The value of information for an* *investor with utility from terminal wealth only is given by*

$$z = 1 - \sqrt{\frac{\gamma - \frac{v(0)T}{1+v(0)T}}{\gamma - \frac{v_\theta(0)T}{1+v_\theta(0)T}}}.$$

*Proof.* Substituting the indirect utilities from Proposition 4.2 and Proposition 5.3 into a rearranged version of (28) gives

$$\begin{aligned}
(1-z)^{\gamma-1} &= \left( \frac{1+(1-\frac{1}{\gamma})vT}{[1+vT]^{1-\frac{1}{\gamma}}} \frac{[1+v_\theta(0)T]^{1-\frac{1}{\gamma}}}{1+(1-\frac{1}{\gamma})v_\theta(0)T} \right)^{\frac{\gamma}{2}} \sqrt{\frac{1+(1-\frac{1}{\gamma})v_\theta(0)T}{1+(1-\frac{1}{\gamma})vT}} \\
&= \left( \frac{[1+v_\theta(0)T]^{1-\frac{1}{\gamma}}}{[1+vT]^{1-\frac{1}{\gamma}}} \frac{1+(1-\frac{1}{\gamma})vT}{1+(1-\frac{1}{\gamma})v_\theta(0)T} \right)^{\frac{\gamma}{2}} \sqrt{\frac{1+(1-\frac{1}{\gamma})v_\theta(0)T}{1+(1-\frac{1}{\gamma})vT}} \\
&= \left( \frac{1+v_\theta(0)T}{1+vT} \right)^{\frac{\gamma-1}{2}} \left( \frac{1+(1-\frac{1}{\gamma})vT}{1+(1-\frac{1}{\gamma})v_\theta(0)T} \right)^{\frac{\gamma-1}{2}}.
\end{aligned}$$

This immediately reduces to

$$\begin{aligned}
1-z &= \sqrt{\frac{1+v_\theta(0)T}{1+vT} \frac{1+(1-\frac{1}{\gamma})vT}{1+(1-\frac{1}{\gamma})v_\theta(0)T}} \\
&= \sqrt{\frac{1+(1-\frac{1}{\gamma})vT}{1+vT} \frac{1+v_\theta(0)T}{1+(1-\frac{1}{\gamma})v_\theta(0)T}} \\
&= \sqrt{\frac{\gamma - \frac{vT}{1+vT}}{\gamma - \frac{v_\theta(0)T}{1+v_\theta(0)T}}},
\end{aligned}$$

from which the claim follows. □

**Remark.** Recalling that the conditional prior variance $v_\theta(0) = (1-\rho^2)v$, the value of information can be expressed as

$$z = 1 - \sqrt{\frac{\gamma - \frac{vT}{1+vT}}{\gamma - \frac{(1-\rho^2)vT}{1+(1-\rho^2)vT}}} \in [0,1].$$

When the signal is uncorrelated with the unobservable market price of risk ($\rho = 0$), investors are not willing to pay anything since the information is useless. On the other hand, the fraction of wealth investors are willing to sacrifice for a signal perfectly correlated with the unobservable market price of risk is bounded by

$$z = 1 - \sqrt{1 - \frac{1}{\gamma}\frac{vT}{1 + vT}}.$$

**Remark** (Utility from intermediate consumption)**.** As the indirect utility function for an investor with utility from intermediate consumption and possibly terminal wealth cannot be found analytically, the value of private information must be computed by direct application of (28) to the numerical values of the indirect utility functions involved.

**Example 5.7.** In this example the certainty equivalent is studied in more detail. Figures 3–5 illustrate that the fraction of wealth an investor is willing to pay for a signal about the expected stock return depends on the risk aversion coefficient as well as the standard deviation of the noise contained in the signal. The certainty equivalent is decreasing in both dimensions. The lower the standard deviation the more precise the signal is. In that case, investors are able to make more qualified investment decisions thereby obtaining a higher utility, which explains the increasing certainty equivalent. That investors become more reluctant to obtain better information when they are more risk averse might seem counterintuitive at first. Remember, however, that the investor allocates a smaller fraction of wealth to the stock, when risk aversion is high. Since the investor's wealth is less exposed to fluctuations in the stock price, more information is not as badly needed leading to a lower certainty equivalent.

An investor deriving utility from terminal wealth only is willing to pay a larger fraction of wealth for a signal than an investor who derives utility from intermediate consumption. Since the former investor has no consumption smoothing motives it is relatively more important that he makes a good investment decision. The latter investor's consumption pattern allows for consumption to be smoothed over time, makes the overall strategy less sensitive to the stock return over the full investment horizon. A conclusion to be drawn from this analysis is that institutional investors should be more willing to gather information about expected stock returns than the conventional investor.

Finally, the longer the investment horizon the larger a fraction of wealth an investor is willing to pay to become informed. This is expected, as more information implies better investment decisions over a longer time span.

# 6 Signal about terminal price

Now, consider the problem faced by an investor who receives a signal about the terminal stock price $S_T$ or equivalently $\bar{B}_T$. The signal is on the form

$$\bar{\psi} = \bar{B}_T + \bar{\varepsilon} = B_T + \theta T + \bar{\varepsilon},$$

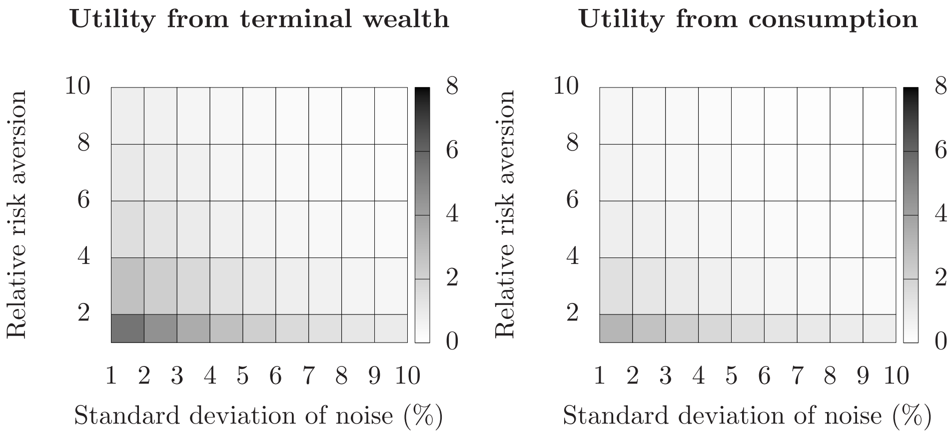


Figure 3: Certainty equivalent with an investment horizon of $T = 5$ years.

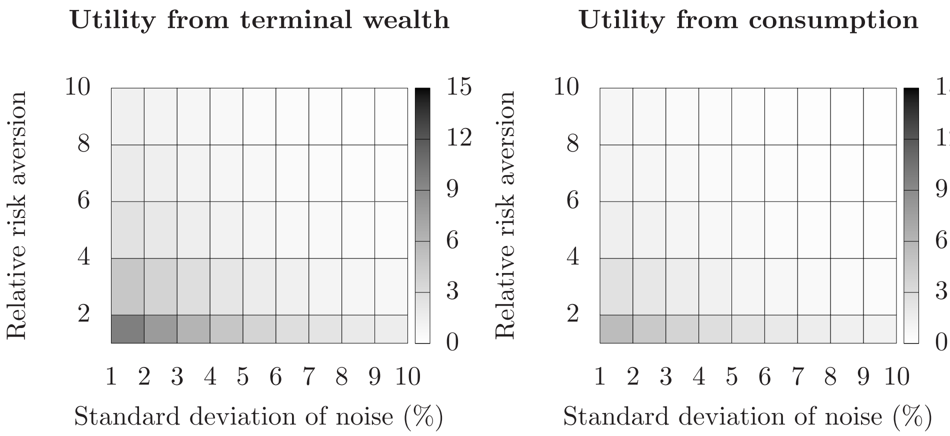


Figure 4: Certainty equivalent with an investment horizon of $T = 10$ years.

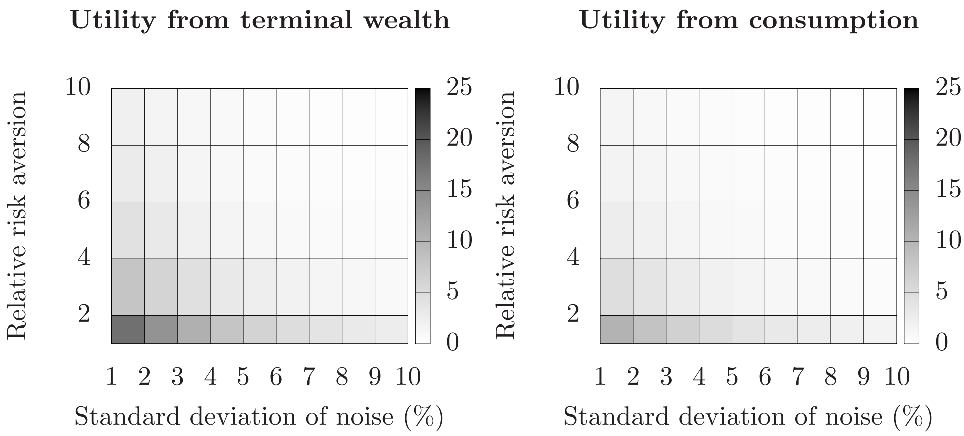


Figure 5: Certainty equivalent with an investment horizon of $T = 25$ years.

where $\bar{\varepsilon} \sim \mathcal{N}(0, \sigma_{\bar{\varepsilon}}^2)$ is assumed independent of $\mathcal{F}_t^B$ and $\theta$. Since the signal is correlated with the unobservable market price of risk, it provides the investor with more information, based on which a more qualified estimate of the market price of risk can be carried out. The filtering problem can be stated as

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observations} & d\bar{B}_t = \theta dt + dB_t & \bar{B}_0 = 0 \\ & d\psi = 0 & \psi_0 = \bar{\psi} = \bar{B}_T + \bar{\varepsilon} \end{array}$$

Since $\bar{\psi}$ is dependent on $B_T$, $B_t$ is not a Brownian motion in the enlarged filtration $\mathcal{F}_t^\psi$. The first step is to rewrite the filtration problem in terms of a $\mathcal{F}_t^\psi$-Brownian motion. In the construction of this Brownian motion, the following Lemma is applied.

**Lemma 6.1.** *Let $(X_t)$ and $(Y_t)$ be independent stochastic processes. Then*

$$\mathbb{E}\left[X_t \,\middle|\, \mathcal{F}_u^X \vee \mathcal{F}_u^Y\right] = \mathbb{E}\left[X_t \,\middle|\, \mathcal{F}_u^X\right].$$

*Proof.* Since $\{A \cap B \,|\, A \in \mathcal{F}_u^X, B \in \mathcal{F}_u^Y\}$ generates $\mathcal{F}_u^X \vee \mathcal{F}_u^Y$, the result follows from

$$\begin{aligned}
\int_{A\cap B} X_t d\mathbb{P} &= \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_B X_t\right] \\
&= \mathbb{E}\left[\mathbf{1}_A X_t\right] \mathbb{E}\left[\mathbf{1}_B\right] && \text{by independence} \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_A X_t \,|\, \mathcal{F}_u^X\right]\right] \mathbb{E}\left[\mathbf{1}_B\right] && \text{by iterated expectations} \\
&= \mathbb{E}\left[\mathbf{1}_A \mathbb{E}\left[X_t \,|\, \mathcal{F}_u^X\right]\right] \mathbb{E}\left[\mathbf{1}_B\right] && \text{since } A \in \mathcal{F}_u^X \\
&= \mathbb{E}\left[\mathbf{1}_A \mathbf{1}_B \mathbb{E}\left[X_t \,|\, \mathcal{F}_u^X\right]\right] && \text{by independence} \\
&= \int_{A\cap B} \mathbb{E}\left[X_t \,|\, \mathcal{F}_u^X\right] d\mathbb{P}.
\end{aligned}$$

□

**Lemma 6.2.** *The process* $(\tilde{B}_t)$ *defined by*

$$\tilde{B}_t \triangleq B_t - \int_0^t \frac{\bar{\psi} - B_s - \theta T}{T - s + \sigma_{\bar{\varepsilon}}^2} ds$$

*is a* $(\mathbb{P}, \mathcal{F}_t^\psi)$*-Brownian motion.*

*Proof.* To prove that $(\tilde{B}_t)$ is a $(\mathbb{P}, \mathcal{F}_t^\psi)$-martingale it is first noticed that

$$\begin{aligned}
\mathbb{E}\left[\tilde{B}_t - \tilde{B}_u \,\middle|\, \mathcal{F}_u^\psi\right] &= \mathbb{E}\left[B_t - B_u + \int_u^t \frac{\bar{\psi} - B_s - \theta T}{T - s + \sigma_{\bar{\varepsilon}}^2} ds \,\middle|\, \mathcal{F}_u^\psi\right] \\
&= \mathbb{E}\left[-(\bar{\psi} - B_t) + (\bar{\psi} - B_u) + \int_u^t \frac{\bar{\psi} - B_s - \theta T}{T - s + \sigma_{\bar{\varepsilon}}^2} ds \,\middle|\, \mathcal{F}_u^\psi\right].
\end{aligned}$$

In this expression

$$\begin{aligned}
\mathbb{E}\left[\bar{\psi} - B_t \,|\, \mathcal{F}_u^\psi\right] &= \mathbb{E}\left[\bar{\psi} - B_t \,|\, \bar{\psi} - B_u\right] |_{x=\theta} \\
&= xT + \frac{T - t + \sigma_{\bar{\varepsilon}}^2}{T + \sigma_{\bar{\varepsilon}}^2}(\bar{\psi} - B_u - xT)\bigg|_{x=\theta} \\
&= \theta T + \frac{T - t + \sigma_{\bar{\varepsilon}}^2}{T + \sigma_{\bar{\varepsilon}}^2}(\bar{\psi} - B_u - \theta T),
\end{aligned}$$

where Lemma 6.1 and Liptser and Shiryaev (2001b, Theorem 13.1) have been applied to obtain the first and second equalities respectively. Hence

$$\begin{aligned}
\mathbb{E}\left[\tilde{B}_t - \tilde{B}_u \,\middle|\, \mathcal{F}_u^\psi\right] &= \frac{t-u}{T + \sigma_{\bar{\varepsilon}}^2}(\bar{\psi} - B_u - \theta T) + \int_u^t \frac{\bar{\psi} - B_u - \theta T}{T + \sigma_{\bar{\varepsilon}}^2} ds \\
&= 0,
\end{aligned}$$

which along with $\langle \hat{B}, \hat{B} \rangle_t = t$ show that $(\hat{B}_t)$ is a $(\mathbb{P}, \mathcal{F}_t^\psi)$-Brownian motion by the Levy characterization. □

As a consequence of Lemma 6.2, the filtering problem can be expressed as

$$\begin{array}{lll} \text{system} & d\theta = 0 & \theta \sim \mathcal{N}(m, v) \\ \text{observations} & d\bar{B}_t = f(t)\left(\sigma_{\bar{\varepsilon}}^2\theta + \bar{\psi} - \bar{B}_t\right)dt + d\tilde{B}_t & \bar{B}_0 = 0 \\ & d\psi = 0 & \psi_0 = \bar{\psi} = \bar{B}_T + \bar{\varepsilon} \end{array}$$

where $f(t) = (T - t + \sigma_{\bar{\varepsilon}}^2)^{-1}$. For later use, it is noted that $f'(t) = f(t)^2$. The above filtering problem fits into (1)–(3) with $a_\theta(t) = f(t)\sigma_{\bar{\varepsilon}}^2$, $\bar{a}(t) = -f(t)$, and $a_\psi(t) = f(t)$. Since $\tilde{B}_t$ is a Brownian motion with respect to the filtration $\mathcal{F}_t^\psi$ the following Corollary of Theorem 2.1 is easily proven.

**Corollary 6.3.** *The solution of the filtering problem satisfies the system of equations*

$$\begin{aligned} d\hat{\theta}_t &= \bar{v}(t)f(t)\sigma_{\bar{\varepsilon}}^2\left[d\bar{B}_t - f(t)\left(\sigma_{\bar{\varepsilon}}^2\hat{\theta}_t + \bar{\psi} - \bar{B}_t\right)dt\right] & \hat{\theta}_0 &= \mathbb{E}[\theta \mid \bar{\psi}] \\ \bar{v}'(t) &= -\left[\bar{v}(t)f(t)\sigma_{\bar{\varepsilon}}^2\right]^2 & \bar{v}(0) &= \mathbb{E}[(\theta - \hat{\theta}_0)^2 \mid \bar{\psi}]. \end{aligned}$$

*Proof.* Resembles the proof of Corollary 4.1 and is omitted. □

By Lemma 2.2 the innovation process $\hat{B}_t \triangleq \bar{B}_t - \int_0^t f(s)(\sigma_{\bar{\varepsilon}}^2\hat{\theta}_s + \bar{\psi} - \bar{B}_s)ds$ is a $(\mathbb{P}, \bar{\mathcal{F}}_t^\psi)$-Brownian motion. Noticing further that the market price of risk process perceived by the investor $X_t = a_\theta(t)\hat{\theta}_t + \bar{a}(t)\bar{B}_t + a_\psi(t)\psi = f(t)(\sigma_{\bar{\varepsilon}}^2\hat{\theta}_t + \bar{\psi} - \bar{B}_t)$ has dynamics given by

$$\begin{aligned} dX_t &= f'(t)(\sigma_{\bar{\varepsilon}}^2\hat{\theta}_t + \bar{\psi} - \bar{B}_t)dt + f(t)\left[\sigma_{\bar{\varepsilon}}^2 d\hat{\theta}_t - d\bar{B}_t\right] \\ &= f(t)X_t dt + f(t)\left[\sigma_{\bar{\varepsilon}}^2 d\hat{\theta}_t - d\bar{B}_t\right] \\ &= f(t)\sigma_{\bar{\varepsilon}}^2\bar{v}(t)f(t)\sigma_{\bar{\varepsilon}}^2 d\hat{B}_t - f(t)d\hat{B}_t \\ &= f(t)\left(\sigma_{\bar{\varepsilon}}^4 f(t)\bar{v}(t) - 1\right)d\hat{B}_t \\ &= \tilde{v}(t)d\hat{B}_t, \end{aligned}$$

in which $\tilde{v}(t) = f(t)\left(\sigma_{\bar{\varepsilon}}^4 f(t)\bar{v}(t) - 1\right)$ satisfies

$$\begin{aligned} \tilde{v}'(t) &= f'(t)\left(\sigma_{\bar{\varepsilon}}^4 f(t)\bar{v}(t) - 1\right) + f(t)\sigma_{\bar{\varepsilon}}^4\left(f'(t)\bar{v}(t) + f(t)\bar{v}'(t)\right) \\ &= f(t)\tilde{v}(t) + f(t)^2\sigma_{\bar{\varepsilon}}^4\left(f(t)\bar{v}(t) - f(t)^2\bar{v}(t)^2\sigma_{\bar{\varepsilon}}^4\right) \\ &= f(t)\tilde{v}(t) - f(t)^2\bar{v}(t)\sigma_{\bar{\varepsilon}}^4\tilde{v}(t) \\ &= f(t)\left(1 - f(t)\bar{v}(t)\sigma_{\bar{\varepsilon}}^4\right)\tilde{v}(t) \\ &= -\tilde{v}(t)^2. \end{aligned}$$

Hence the perceived optimization problem is analogous to the problem of the generic investor with $\hat{v}(t) = \tilde{v}(t)$. Taking this difference into account, all of the proposition from Section 3 holds for an investor receiving a signal about the terminal stock price. This observation leads to the following information equivalence result.

**Proposition 6.4.** *When* $v_\theta(0) = f(0)\left(\sigma_{\bar{\varepsilon}}^4 f(0)\bar{v}(0) - 1\right)$ *a signal about the market price of risk on the form* $\psi_\theta = \theta + \bar{\varepsilon}_\theta$ *is equivalent to a signal about the terminal stock price on the form* $\bar{\psi} = \bar{B}_T + \bar{\varepsilon}$.

A natural question to ask is whether the condition in the proposition has an economic meaning. Interpreting $\tilde{v}(0) = f(0)\left(\sigma_{\bar{\varepsilon}}^4 f(0)\bar{v}(0) - 1\right)$ as the initial instantaneous volatility of the market price of risk perceived by an investor receiving a signal about the terminal stock price, the condition $v_\theta(0) = \tilde{v}(0)$ states that the sensitivity of the perceived market prices of risk to the shocks perceived by the two types of investors must be the same. This is for example the case when the variances of $\log S_T$ given the two signals are equal.

# 7 Future research

The first item on the agenda is to study examples where the two types of signals lead to different conclusions. This could lead to more interesting predictions such as which signal is better for different types of investors. Also, the effects on portfolio choice should be studied.

The preceding analysis was restricted to two very simple types of signals. Having a more general information technology such as e.g. Corcuera, Imkeller, Kohatsu-Higa, and Nualart (2004) would allow for more interesting effects.

Having analyzed the value of information in a partial equilibrium, it could be of great interest to study a general equilibrium model with information acquisition determined endogenously. Since the financial market partly reveals the private knowledge of investors through their demand of assets the equilibrium framework chosen must not be fully revealing. Since asset prices reveal some information one can think of the introduction of a new asset to the market as a mean to giving investors more information about existing assets. Hence it is possible to study financial innovation within such a model. Earlier work in this direction include Cao (1999) and Massa (2002) but both papers equip investor with exponential utility in order to be able to work things out.

# 8 Conclusion

In a financial market with incomplete information about the expected stock return the value of having private information about either the expected stock return or the terminal stock price was derived. The solution to the optimization problem of an informed investor turned out to be structurally very similar to the problem of an uninformed investor even though the two types of signals studied could give very different results. A condition for them to give the same solution was provided.

Through examples the effect of having parameter uncertainty and the value of receiving a private signal were studied. Parameter uncertainty imply a hedging/speculative behavior depending on the risk aversion of the investor. The value of having private information is increasing in the investment horizon and the precision of the signal, but decreasing in the risk aversion coefficient of the investor. The first two effects are quite intuitive. First, it must be more valuable to be able to exploit private information over a longer time horizon. Second, the better the information the more qualified investments should an investor be able to make. That the value of private information is decreasing in the risk aversion is due to the investor placing a smaller fraction of wealth in the stock and the signal therefore becomes less important.

# References


Amendinger, J., D. Becherer, and M. Schweizer (2003). A monetary value for initial information in portfolio optimization. *Finance and Stochastics 7*(1), 29–46.

Amendinger, J., P. Imkeller, and M. Schweizer (1998). Additional logarithmic utility of an insider. *Stochastic Processes and their Applications 75*(2), 263–286.

Benarous, J.-D., L. Mazliak, and R. Rouge (2000). Filtering and control with information increasing. *Methodology and Computing in Applied Probability 2*(2), 123–135.

Brennan, M. J. (1998). The role of learning in dynamic portfolio decisions. *European Finance Review 1*(3), 295–306.

Brennan, M. J. and Y. Xia (2001). Assessing asset pricing anomalies. *Review of Financial Studies 14*(4), 905–942.

Cao, H. H. (1999). The effect of derivative assets on information acquisition and price behavior in a rational expectations equilibrium. *Review of Financial Studies 12*(1), 131–163.

Corcuera, J. M., P. Imkeller, A. Kohatsu-Higa, and D. Nualart (2004). Additional utility of insiders with imperfect dynamical information. *Finance and Stochastics 8*(3), 437–450.

Cvitanić, J., A. Lazrak, L. Martellini, and F. Zapatero (2006). Dynamic portfolio choice with parameter uncertainty and the economic value of analysts' recommendations. *Review of Financial Studies 19*(4), 1113–1156.

David, A. (1997). Fluctuating confidence in stock markets: Implications for returns and volatility. *The Journal of Financial and Quantitative Analysis 32*(4), 427–462.

Detemple, J. B. (1986). Asset pricing in a production economy with incomplete information. *The Journal of Finance 41*(2), 383–391.

Dothan, M. U. and D. Feldman (1986). Equilibrium interest rates and multiperiod bonds in a partially observable economy. *The Journal of Finance 41*(2), 369–382.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (Third ed.). Princeton University Press.

Gennotte, G. (1986). Optimal portfolio choice under incomplete information. *The Journal of Finance 41*(3), 733–749.

Hansen, S. L. (2006). *Three Topics in Asset Allocation and Asset Pricing*. Ph. D. thesis, University of Southern Denmark.

Honda, T. (2003). Optimal portfolio choice for unobservable and regime-switching mean returns. *Journal of Economic Dynamics and Control 28*(1), 45–78.

Lakner, P. (1998). Optimal trading strategy for an investor: the case of partial information. *Stochastic Processes and their Applications 76*(1), 77–97.

Liptser, R. S. and A. N. Shiryaev (2001a). *Statistics of Random Processes*, Volume I. General Theory. Springer.

Liptser, R. S. and A. N. Shiryaev (2001b). *Statistics of Random Processes*, Volume II. Applications. Springer.

Liu, J., E. Peleg, and A. Subrahmanyam (2005, October). Long-lived private information in a continuous time economy: Portfolio choice, optimal consumption, and utility gain. Working paper, UCLA.

Massa, M. (2002). Financial innovation and information: The role of derivatives when a market for information exists. *Review of Financial Studies 15*(3), 927–957.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics 51*(3), 247–257. Reprinted as Chapter 4 in Merton (1992).

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory 3*(4), 373–413. Erratum: Merton (1973). Reprinted as Chapter 5 in Merton (1992).

Merton, R. C. (1973). Erratum. *Journal of Economic Theory 6*(2), 213–214.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics 8*(4), 323–361.

Merton, R. C. (1992). *Continuous-Time Finance*. Blackwell Publishers Inc.

Pikovsky, I. and I. Karatzas (1996). Anticipative portfolio optimization. *Advances in Applied Probability 28*(4), 1095–1122.

Rogers, L. C. G. (2001). The relaxed investor and parameter uncertainty. *Finance and Stochastics 5*(2), 131–154.

Samuelson, P. A. (1991). Long-run risk tolerance when equity returns are mean reregressing: Pseudoparadoxes and vindication of "businessman's risk". In W. C. Brainard, W. D. Nordhaus, and H. W. Watts (Eds.), *Money, Macroeconomics, and Economic Policy*, pp. 181–200. MIT Press.

Veronesi, P. (1999). Stock market overreactions to bad news in good times: a rational expectations equilibrium model. *Review of Financial Studies 12*(5), 975–1007.

Wachter, J. A. (2002, March). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *The Journal of Financial and Quantitative Analysis 37*(1), 63–91.

Xia, Y. (2001). Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. *The Journal of Finance 56*(1), 205–246.