

## The semireactive bargaining set of a cooperative game\*

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**Abstract.** The semireactive bargaining set, a solution for cooperative games, is introduced. This solution is in general a subsolution of the bargaining set and a supersolution of the reactive bargaining set. However, on various classes of transferable utility games the semireactive and the reactive bargaining set coincide. The semireactive prebargaining set on TU games can be axiomatized by one-person rationality, the reduced game property, a weak version of the converse reduced game property with respect to subgrand coalitions, and subgrand stability. Furthermore, it is shown that there is a suitable weakening of subgrand stability, which allows to characterize the prebargaining set. Replacing the reduced game by the imputation saving reduced game and employing individual rationality as an additional axiom yields characterizations of both, the bargaining set and the semireactive bargaining set.

**Key words:** TU game, bargaining set

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### 0. Introduction

The semireactive bargaining set is a set-valued solution of cooperative transferable utility games. Its definition is strongly related to the definition (see Aumann and Maschler (1964)) of the bargaining set  $\mathcal{M} = \mathcal{M}_1^{(i)}$ . The only difference between the definitions of the two bargaining sets is that in the classical one the “objector” (player  $k$  objects against some “partner”  $l$ ) has to announce his objection in advance, whereas in the definition of the semireactive bargaining set the objector only has to announce in advance the coalition which he plans to object with. The “complete” objection is announced ex post, i.e., after

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his partner has already announced the coalition which he will try to counter object with (if there is one). In view of the fact that the objector may react to the coalition announced by player  $l$ , the semireactive bargaining set is a subset of the classical bargaining set. In the definition of the reactive bargaining set, introduced by Granot (1994), the objector is allowed to wait with his objection until his partner has announced his defending coalition. Therefore we use the expression “semireactive”. Indeed, the semireactive bargaining set contains the reactive bargaining set and is contained in the classical bargaining set.

Though in general larger than the reactive bargaining set, the semireactive bargaining set is easier to compute than the classical one. Moreover, it coincides with the reactive bargaining set for various classes of games. Like the mentioned well-known bargaining sets the notion of the semireactive bargaining set can be extended to nontransferable utility games. One of the main results (Theorem 3.1) shows that the semireactive prebargaining set has an axiomatization that is similar to Peleg’s (1986) axiomatization of the prekernel. Moreover, a suitable modification of this axiomatization also characterizes the semireactive bargaining set.

The paper is organized as follows: In Section 1 the notation and some definitions are presented. Moreover, it is shown that the semireactive bargaining set can be described as a finite union of polytopes. This description can be used to calculate the semireactive (pre)bargaining set of a generic TU game.

In Section 2 it turns out that the semireactive prebargaining set satisfies anonymity, covariance under strategic equivalence, and the reduced game property with respect to “Davis-Maschler” (1965) reduced games. *Subgrand stability*, a property which excludes the possibility that *subgrand* coalitions (coalitions that contain all but one player) can be used in a justified objection, is introduced and it is shown that the semireactive prebargaining set satisfies an appropriate version of the converse reduced game property. If the Davis-Maschler reduced game is replaced by the imputations saving reduced game in the sense of Snijders (1995), then the semireactive bargaining set is shown to satisfy both, the reduced game property and a suitable version of the converse reduced game property. Indeed, this converse reduced game property requires subgrand stable imputations.

Section 3 is devoted to axiomatize the semireactive **prebargaining** set. In fact, one-person rationality, subgrand stability, the reduced game property, and the appropriate version of the converse reduced game property are logically independent axioms which determine the semireactive prebargaining set. If the Davis-Maschler reduced game is replaced by the Snijders reduced game, then the semireactive bargaining set can be axiomatized analogously, when individual rationality is added to the axioms. The axiomatization of the semireactive **prebargaining** set is similar to the well-known axiomatization (see Peleg (1986)) of the **prekernel**. However, the axiomatization of the semireactive bargaining set does not lead to an analogous characterization of the kernel.

Section 4 shows that the semireactive prebargaining set satisfies *reasonableness*, thus the *nullplayer property*, if the considered games are superadditive. For superadditive simple games it turns out that the semireactive bargaining set coincides with the *positive prekernel* (see Sudhölter and Peleg (2000)) and, thus, with the reactive bargaining set.

In Section 5 it is shown that subgrand stability can be weakened in such a way that it can be used to characterize the classical (pre)bargaining set. We admit that this weaker property is also less intuitive.

Section 6 briefly describes how the definition of the semireactive (pre)-bargaining set can be extended to TU games with coalition structures and to NTU games. Moreover, a set-valued dynamic system is presented which leads to the semireactive (pre)bargaining set and which can be seen as the suitable analogue to the one leading to the reactive bargaining set (see Granot and Maschler (1997)).

## 1. Notation and definitions

Let  $U$  be a set (the *universe of players*). A *cooperative game with transferable utility* (a TU game) – a *game* – is a pair  $(N, v)$ , where  $N$  is a finite nonvoid subset of  $U$  and  $v : 2^N \rightarrow \mathbb{R}$ ,  $v(\emptyset) = 0$  is a mapping (the *coalitional function*). Here  $2^N = \{S \subseteq N\}$  is the set of *coalitions* of  $(N, v)$ . Let  $\Gamma_U$  denote the set of all games.

The set of *feasible payoff vectors* of a game  $(N, v)$  is denoted by

$$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

whereas

$$\mathcal{I}^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

is the set of *preimputations* of  $(N, v)$  and

$$\mathcal{I}(N, v) = \{x \in \mathcal{I}^*(N, v) \mid x_k \geq v(\{k\}) \forall k \in N\}$$

is the set of *individually rational preimputations (imputations)* of  $(N, v)$ . Here

$$x(S) = \sum_{i \in S} x_i \quad (x(\emptyset) = 0)$$

for each  $x \in \mathbb{R}^N$  and  $S \subseteq N$ . Additionally, let  $x_S$  denote the restriction of  $x$  to  $S$ , i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S.$$

For disjoint coalitions  $S, T \subseteq N$  and  $x \in \mathbb{R}^N$  let  $(x_S, x_T) = x_{S \cup T}$ . Let  $\Gamma_U^{\mathcal{I}} = \{(N, v) \in \Gamma_U \mid \sum_{i \in N} v(\{i\}) \leq v(N)\}$  denote the set of games which possess non-empty sets of imputations.

A *solution*  $\sigma$  on a set  $\Gamma$  of games is a mapping that assigns a set  $\sigma(N, v) \subseteq X(N, v)$  to every game  $(N, v) \in \Gamma$ .

If  $\Gamma$  is not specified, then  $\sigma$  is a solution on  $\Gamma_U$ .

Let  $(N, v)$  be a game,  $x \in \mathbb{R}^N$ , and  $k, l \in N$  be distinct players. Define the collection  $\mathcal{T}_{kl}(N)$  by

$$\mathcal{T}_{kl} = \mathcal{T}_{kl}(N) = \{S \subseteq N \setminus \{l\} \mid k \in S\}.$$

Hence,  $\mathcal{T}_{kl}$  is the set of coalitions containing  $k$  and not containing  $l$ . An *objection* of  $k$  against  $l$  at  $x$  (w.r.t.  $(N, v)$ ) is a pair  $(P, y)$  satisfying

$$P \in \mathcal{T}_{kl}, y \in \mathbb{R}^P, y(P) = v(P), \quad \text{and} \quad y \gg x_P \text{ (i.e., } y_i > x_i \forall i \in P).$$

If  $(P, y)$  has these properties, then we say that  $k$  is able to *object* against  $l$  via coalition  $P$ . Note that  $k$  is able to object against  $l$  via  $S \in \mathcal{T}_{kl}$ , if and only if the *excess*  $e(S, x, v) = v(S) - x(S)$  is strictly positive.

A *counter objection* to an objection  $(P, y)$  of  $k$  against  $l$  at  $x$  is a pair  $(Q, z)$  satisfying

$$Q \in \mathcal{T}_{lk}, z \in \mathbb{R}^Q, z(Q) = v(Q), z \geq x_Q, \quad \text{and} \quad z_{P \cap Q} \geq y_{P \cap Q}.$$

If  $(Q, z)$  has these properties, then we say that  $l$  is able to *counter*  $(P, y)$  via coalition  $Q$ . Note that  $l$  can counter  $(P, y)$  via  $Q \in \mathcal{T}_{lk}$ , if and only if  $e(Q, x, v) \geq y(P \cap Q) - x(P \cap Q)$ .

**Definition 1.1.** The **semireactive prebargaining set**  $\mathcal{M}_{\text{sr}}^*(N, v)$  of a game  $(N, v)$  is the set of all preimputations  $x \in \mathcal{I}^*(N, v)$  that satisfy the following condition for any pair of distinct players  $(k, l) \in N \times N$  and for any  $P \in \mathcal{T}_{kl}$ :

There is  $Q \in \mathcal{T}_{lk}$  such that any objection of  $k$  against  $l$  via  $P$  can be countered by  $l$  via  $Q$ .

The **semireactive bargaining set** of  $(N, v)$  is defined to be the set

$$\mathcal{M}_{\text{sr}}(N, v) = \mathcal{M}_{\text{sr}}^*(N, v) \cap \mathcal{I}(N, v)$$

of individually rational elements of the semireactive prebargaining set.

Let  $(N, v)$  be a game and  $x \in \mathbb{R}^N$ . In order to compare the definitions of the ‘‘classical’’ bargaining set, the reactive bargaining set, and the semireactive bargaining set, we define three relations  $\preceq = \preceq^{N, v, x}$ ,  $\preceq_r = \preceq_r^{N, v, x}$ , and  $\preceq_{\text{sr}} = \preceq_{\text{sr}}^{N, v, x}$  on  $N$ .

(1)  $k \preceq l$ , if:

$$\begin{aligned} &\forall P \in \mathcal{T}_{kl} \text{ with } e(P, x, v) > 0 \quad \text{and} \\ &\forall y \in \mathbb{R}^P \text{ with } y(P) = v(P), \quad y \gg x_P \\ &\exists Q \in \mathcal{T}_{lk} \text{ such that} \\ &\exists z \in \mathbb{R}^Q \text{ with } z(Q) = v(Q), \quad z \geq x_Q, \quad z_{P \cap Q} \geq y_{P \cap Q} \end{aligned} \tag{1.1}$$

(2)  $k \preceq_r l$ , if

$$\begin{aligned} &\exists Q \in \mathcal{T}_{lk} \text{ such that} \\ &\forall P \in \mathcal{T}_{kl} \text{ with } e(P, x, v) > 0 \quad \text{and} \\ &\forall y \in \mathbb{R}^P \text{ with } y(P) = v(P), \quad y \gg x_P \\ &\exists z \in \mathbb{R}^Q \text{ with } z(Q) = v(Q), \quad z \geq x_Q, \quad z_{P \cap Q} \geq y_{P \cap Q} \end{aligned} \tag{1.2}$$

(3)  $k \preceq_{\text{sr}} l$ , if

$$\begin{aligned}
 & \forall P \in \mathcal{F}_{kl} \text{ with } e(P, x, v) > 0 \\
 & \exists Q \in \mathcal{F}_{lk} \text{ such that} \\
 & \forall y \in \mathbb{R}^P \text{ with } y(P) = v(P), \quad y \gg x_P \\
 & \exists z \in \mathbb{R}^Q \text{ with } z(Q) = v(Q), \quad z \geq x_Q, \quad z_{P \cap Q} \geq y_{P \cap Q}
 \end{aligned} \tag{1.3}$$

Let  $\succ$ ,  $\succ_r$ , and  $\succ_{\text{sr}}$  be the negations of  $\preceq$ ,  $\preceq_r$ , and  $\preceq_{\text{sr}}$ . We shall say that player  $k$  has a *justified objection* against player  $l$  at  $x$  in the sense of the bargaining set, **reactive** bargaining set, or **semireactive** bargaining set, respectively, if  $k \succ l$ ,  $k \succ_r l$ , or  $k \succ_{\text{sr}} l$ , respectively.

The *prebargaining set*  $\mathcal{M}^*(N, v)$  (see Aumann and Maschler (1964)) and the *reactive prebargaining set*  $\mathcal{M}_r^*(N, v)$  (see Granot (1994)) is the set of all pre-imputations such that no player has a justified objection against any other player in the sense of the prebargaining set or reactive prebargaining set respectively. Note that the condition leading to the reactive prebargaining set (1.2) arises from the condition leading to the prebargaining set (1.1) by exchanging the order of **two** quantifiers. In fact (1.2) arises from (1.1) by exchanging the first and the third row. The *bargaining set*  $\mathcal{M}(N, v)$  and the *reactive bargaining set*  $\mathcal{M}_r(N, v)$  arise from the corresponding prebargaining sets by their intersection with the set of imputations.

A different change of the order (see 1.3) leads to the semireactive (pre)-bargaining set (see Definition 1.1). In fact (1.3) arises from (1.1) by exchanging the second and the third row. In view of the fact that

$$k \preceq_r l \Rightarrow k \preceq_{\text{sr}} l \Rightarrow k \preceq l$$

we obtain

$$\mathcal{M}_r^*(N, v) \subseteq \mathcal{M}_{\text{sr}}^*(N, v) \subseteq \mathcal{M}^*(N, v)$$

as well as

$$\mathcal{M}_r(N, v) \subseteq \mathcal{M}_{\text{sr}}(N, v) \subseteq \mathcal{M}(N, v).$$

The mentioned prebargaining sets are nonempty and the mentioned bargaining sets are nonempty provided that the set of imputations is nonempty. Indeed, the *prekernel*

$$\mathcal{K}^*(N, v) = \{x \in \mathcal{F}^*(N, v) \mid s_{kl}(x, v) = s_{lk}(x, v) \forall k, l \in N, k \neq l\}$$

is a nonempty set (see Davis and Maschler (1965)). Here  $s_{kl}(x, v) = \max_{S \in \mathcal{F}_{kl}} e(S, x, v)$  denotes the *maximal surplus* of  $k$  over  $l$ . The prekernel of a game is contained in its reactive prebargaining set (see Granot (1994)). Moreover, the *kernel*

$$\mathcal{K}(N, v) = \{x \in \mathcal{F}(N, v) \mid s_{kl}(x, v) \leq s_{lk}(x, v) \text{ or } x_l = v(\{l\}) \forall k, l \in N, k \neq l\}$$

is a nonempty subset of the reactive bargaining set, if  $\mathcal{I}(N, v) \neq \emptyset$ . Also the *core*

$$\mathcal{C}(N, v) = \{x \in \mathcal{I}(N, v) \mid e(S, x, v) \leq 0 \ \forall S \subseteq N\}$$

is a subset of  $\mathcal{M}_r(N, v)$ . Sudhölter and Peleg (2000) introduced a solution which contains both, the prekernel and the core. This solution is the *positive prekernel*

$$\mathcal{K}_+^*(N, v) = \{x \in \mathcal{I}^*(N, v) \mid (s_{kl}(x, v))_+ = (s_{lk}(x, v))_+ \ \forall k, l \in N, k \neq l\},$$

where  $t_+ = \max\{0, t\}$  denotes the positive part of the real number  $t$ . If some player has an objection via some coalition  $P$ , then  $e(P, x, v) > 0$ . If  $x$  is a member of the positive prekernel of a game, then every objection of player  $k$  against player  $l$  can be countered by any coalition attaining the maximal surplus of  $l$  over  $k$ . This fact directly implies

$$\mathcal{C}(N, v) \cup \mathcal{K}_+^*(N, v) \subseteq \mathcal{K}_+^*(N, v) \subseteq \mathcal{M}_r^*(N, v).$$

Similarly it can be shown that the *positive kernel*

$$\begin{aligned} \mathcal{K}_+(N, v) = \{x \in \mathcal{I}(N, v) \mid s_{kl}(x, v) \leq (s_{lk}(x, v))_+ \text{ or} \\ x_l = v(\{l\}) \ \forall k, l \in N, k \neq l\} \end{aligned}$$

contains the core and the kernel and is contained in the reactive bargaining set.

Example 5.3 of Sudhölter and Peleg (2000) presents a game  $(N, v)$  whose core is a singleton and which satisfies

$$\begin{aligned} \mathcal{C}(N, v) \subset \mathcal{K}(N, v) \subset \mathcal{K}_+(N, v), \quad \mathcal{K}_+^* \subset \mathcal{M}_r^*(N, v), \quad \text{and} \\ \mathcal{M}_r(N, v) \subset \mathcal{M}(N, v), \end{aligned}$$

where “ $\subset$ ” means “proper subset”. The following examples show that the reactive bargaining set may be a proper subset of the semireactive bargaining set and that the semireactive bargaining set may be a proper subset of the classical bargaining set. Example 4.4 shows that the positive (pre)kernel may be a proper subset of the reactive bargaining set even if the game is superadditive.

*Example 1.2.* (1) Let  $(N, v)$  be defined by  $N = \{1, 2, 3, 4\}$  and

$$v(S) = \begin{cases} 8, & \text{if } S = N \\ 6, & \text{if } |S| = 3 \text{ or } (|S| = 2 \text{ and } 1 \in S) \\ 5, & \text{if } |S| = 2 \text{ and } 1 \notin S \\ 0, & \text{otherwise} \end{cases}.$$

We shall show that  $x = (2, 2, 2, 2) \in \mathcal{M}_{sr}(N, v) \setminus \mathcal{M}_r(N, v)$ . Note that *interchangeable players* do not have any justified objection against each other in the sense of any bargaining set, as long as they are treated equally. (Players  $k$  and  $l$  are interchangeable, if  $v(S \cup \{k\}) = v(S \cup \{l\}) \ \forall S \subseteq$

$N \setminus \{k, l\}$ .) In our example players 2, 3, 4 are interchangeable and treated equally.

In order to show that  $x$  belongs to the semireactive bargaining set, first observe that  $s_{1k}(x, v) = 2 > s_{k1}(x, v) = 1$  for every  $k \in \{2, 3, 4\}$ , thus every objection of  $k$  against 1 can be countered by a coalition  $Q$  attaining  $s_{1k}(x, v) = e(Q, x, v)$ . Moreover, player 1 can only object against  $k$  via some coalition  $P = \{1, j\}$  for some  $j \in \{2, 3, 4\} \setminus \{k\}$ . Any of these objections can be countered via the complement coalition.

In order to prove that  $x \notin \mathcal{M}_r(N, v)$  we show that 1 has a justified objection against 4 in the sense of the reactive bargaining set. Indeed, it suffices to show that for any  $Q \in \mathcal{T}_{41}$  with  $e(Q, x, v) \geq 0$  there is a coalition  $P \in \mathcal{T}_{14}$  that satisfies  $Q \cap P \neq \emptyset$  and  $e(P, x, v) > e(Q, x, v)$ . If  $Q = \{2, 3, 4\}$ , then  $P = \{1, 2\}$  has the desired properties. If  $Q = \{j, 4\}$  for some  $j \in \{2, 3\}$ , then  $P = \{1, j\}$  has the desired properties.

(2) Let  $(N, v)$  be defined by  $N = \{1, 2, 3, 4\}$  and

$$v(S) = \begin{cases} 6, & \text{if } |S| \geq 3 \\ 5, & \text{if } |S| = 2 \text{ and } 1 \notin S \\ 2, & \text{if } |S| = 2 \text{ and } 1 \in S \\ 0, & \text{otherwise} \end{cases}$$

We shall show that  $x = (0, 2, 2, 2) \in \mathcal{M}(N, v) \setminus \mathcal{M}_{sr}(N, v)$ .

Players 2, 3, 4 are interchangeable and they are treated equally. Moreover, none of them has a justified objection against 1, because  $x_1 = v(\{1\})$ . Player 1 can only object against  $l \in \{2, 3, 4\}$  via  $P = N \setminus \{l\}$ . If  $(P, y)$  is an objection of 1 against  $l$ , then there exists  $j \in P \setminus \{1\}$  with  $y_j < 3$ , thus  $(Q, z)$ , defined by  $Q = \{j, l\}$ ,  $z_l = 2$ ,  $z_j = 3$  is a counter objection.

We show that 1 has a justified objection against 4 in the sense of the semireactive bargaining set. Indeed, let  $P = \{1, 2, 3\}$ . If  $Q = \{j, 4\}$  for some  $j = 2, 3$ , then there is an objection  $(P, y)$  which cannot be countered via  $Q$ , because  $e(Q, x, v) = 1 < 2 = e(P, x, v)$  and  $P \cap Q \neq \emptyset$ .

In what follows we show that the semireactive prebargaining set is a finite union of convex polytopes. To this end we first present an equivalent formulation of (1.3). Let  $(N, v)$  be a game,  $x \in \mathbb{R}^N$ , and  $k, l \in N$  be two distinct players. Then  $k \preceq_{sr} l$  holds true, if and only if the following condition is satisfied:

$$\begin{aligned} \forall P \in \mathcal{T}_{kl} \text{ with } e(P, x, v) > 0 \quad \exists Q \in \mathcal{T}_{lk} \text{ such that} \\ (e(Q, x, v) \geq 0 \text{ and } Q \cap P = \emptyset) \quad \text{or} \quad e(Q, x, v) \geq e(P, x, v) \end{aligned} \tag{1.4}$$

In order to verify the equivalence of (1.4) and (1.3) first assume that (1.3) is valid. Let  $P \in \mathcal{T}_{kl}$  be a coalition with  $e(P, x, v) > 0$  and let  $Q \in \mathcal{T}_{lk}$  be a coalition which satisfies the property required in (1.3). If  $Q \cap P = \emptyset$ , then (1.3) implies the existence of  $z \in \mathbb{R}^Q$  with  $z(Q) = v(Q)$  and  $z \geq x_Q$ , thus  $e(Q, x, v) = z(Q) - x(Q) \geq 0$ . If  $P \cap Q \neq \emptyset$ , then define  $y \in \mathbb{R}^P$  by

$$y_j = \begin{cases} x_j + \varepsilon/|P \setminus Q|, & \text{if } j \in P \setminus Q \\ x_j + (e(P, x, v) - \varepsilon)/|P \cap Q|, & \text{if } j \in P \cap Q \end{cases}$$

Then  $y(P) = v(P)$  and  $y \gg x_P$  whenever  $\varepsilon$  is small enough. Moreover,  $(y - x)(P \cap Q)$  tends to  $e(P, x, v)$ , if  $\varepsilon$  tends to 0. If  $z \in \mathbb{R}^Q$  satisfies  $z(Q) = v(Q)$ ,  $z \geq x_Q$  and  $z_{P \cap Q} \geq y_{P \cap Q}$ , then  $e(Q, x, v) = z(Q) - x(Q) \geq (z - x)(Q \cap P) \geq (y - x)(Q \cap P)$ , thus  $e(Q, x, v) \geq e(P, x, v)$  in this case. To show the opposite direction let  $Q \in \mathcal{T}_{lk}$  be a coalition which has the property required in (1.4) and let  $y \in \mathbb{R}^P$  satisfy  $y(P) = v(P)$  and  $y \gg x_P$ . If  $P \cap Q = \emptyset$ , then  $z \in \mathbb{R}^Q$ , defined by  $z_j = x_j$  for  $j \in Q \setminus \{l\}$  and  $z_l = x_l + e(Q, x, v)$ , is a vector which can be used in (1.3). If  $P \cap Q \neq \emptyset$ , then define  $z$  by

$$z_j = \begin{cases} y_j, & \text{if } j \in P \cap Q \\ x_j, & \text{if } j \in Q \setminus (P \cup \{l\}). \\ x_l + e(Q, x, v) + (x - y)(P \cap Q), & \text{if } j = l \end{cases}$$

Then  $z_l \geq x_l$ , thus  $z$  can be used in (1.3).

In order to show that  $\mathcal{M}_{sr}^*(N, v)$  is a finite union of polyhedral sets, define  $\mathcal{P} = \{(P, k, l) \mid k \in P \subseteq N \setminus \{l\}, l \in N \setminus \{k\}\}$ . Moreover, for  $R, Q \subseteq N$  with  $R \setminus Q \neq \emptyset \neq Q \setminus R$  define the halfspaces

$$\begin{aligned} X^R &= \{x \in \mathcal{S}^*(N, v) \mid e(R, x, v) \leq 0\}, \\ Y^Q &= \{x \in \mathcal{S}^*(N, v) \mid e(Q, x, v) \geq 0\}, \quad \text{and} \\ Z^{R, Q} &= \{x \in \mathcal{S}^*(N, v) \mid e(Q, x, v) \geq e(R, x, v)\}. \end{aligned}$$

Let  $\mathcal{Q}$  denote the collection of all halfplanes of the form  $X^R$ ,  $Y^Q$ , and  $Z^{R, Q}$ . We call a map  $\lambda : \mathcal{P} \rightarrow \mathcal{Q}$  *feasible*, if it satisfies

$$\begin{aligned} \lambda(P, k, l) = X^R &\Rightarrow R = P, \\ \lambda(P, k, l) = Y^Q &\Rightarrow Q \in \mathcal{T}_{lk} \quad \text{and} \quad P \cap Q = \emptyset, \quad \text{and} \\ \lambda(P, k, l) = Z^{R, Q} &\Rightarrow R = P, \quad Q \in \mathcal{T}_{lk} \quad \text{and} \quad P \cap Q \neq \emptyset. \end{aligned}$$

By (1.4) a preimputation  $x$  belongs to  $\mathcal{M}_{sr}^*(N, v)$ , if and only if there is a feasible mapping  $\lambda$  such that

$$x \in A_\lambda := \bigcap_{(P, k, l) \in \mathcal{P}} \lambda(P, k, l).$$

Indeed, if  $x \in \mathcal{M}_{sr}^*(N, v)$ ,  $k, l \in N$  are distinct players, and if  $P \in \mathcal{T}_{kl}$ , then one of the following three cases can occur. (1)  $e(P, x, v) \leq 0$ , i.e.,  $k$  has no objection via  $P$ . In this case define  $\lambda(P, k, l) = X^P$ . (2)  $e(P, x, v) > 0$  and there is a coalition  $Q \in \mathcal{T}_{lk}$  with  $Q \cap P = \emptyset$  and  $e(Q, x, v) \geq 0$ . In this case define  $\lambda(P, k, l) = Y^Q$ . (3)  $e(P, x, v) > 0$  and there is a coalition  $Q \in \mathcal{T}_{lk}$  with  $Q \cap P \neq \emptyset$  and  $e(Q, x, v) \geq e(P, x, v)$ . In this case define  $\lambda(P, k, l) = Z^{P, Q}$ . The definition of  $\lambda$  implies that  $x \in \lambda(P, k, l)$  is true. Conversely, if  $x \in \lambda(P, k, l)$  for some feasible  $\lambda$ , then three cases can occur. (1)  $\lambda(P, k, l) = X^P$ . In this case there is no objection via  $P$ . (2)  $\lambda(P, k, l) = Y^Q$ . In this case every objection of  $k$  against  $l$  can be countered via  $Q$ , because  $Q \cap P = \emptyset$  and  $e(Q, x, v) \geq 0$ . (3)  $\lambda(P, k, l) =$

$Z^P \cdot Q$ . In this case every objection of  $k$  against  $l$  via  $P$  can be countered via  $Q$ , because  $e(Q, x, v) \geq e(P, x, v)$ .

Therefore

$$\mathcal{M}_{\text{sr}}^*(N, v) = \bigcup_{\lambda \text{ is feasible}} A_\lambda \tag{1.5}$$

is shown.

It should be noted that there is a distinguished feasible  $\lambda$  which satisfies  $A_\lambda = \mathcal{C}(N, v)$  (namely the map  $\lambda$  given by  $\lambda(P, k, l) = X^P \forall (P, k, l) \in \mathcal{P}$ ).

**Theorem 1.3.** *If  $(N, v)$  is a game, then  $\mathcal{M}_{\text{sr}}^*(N, v)$  is a finite union of polytopes.*

The following well-known result can be used to prove Theorem 1.3.

**Lemma 1.4.** *If  $(N, v)$  is a game, then  $\mathcal{M}^*(N, v)$  is bounded.*

For the sake of completeness we prove Lemma 1.4 below.

*Proof of Theorem 1.3:* For every feasible mapping  $\lambda$  the set  $A_\lambda$  is the intersection of finitely many halfspaces and, thus, a polyhedral set. There are finitely many feasible maps  $\lambda$ . Therefore the semireactive prebargaining set is a finite union of polyhedral sets by equation (1.5). In view of the fact that the semireactive prebargaining set is contained in the classical prebargaining set, Lemma 1.4 completes the proof. **q.e.d.**

*Proof of Lemma 1.4:* In view of the well-known fact that the prebargaining set satisfies covariance under strategic equivalence (see property (2) of Section 2), we may assume without loss of generality that  $(N, v)$  is *monotonic*, i.e.,  $v(S) \leq v(T)$  for all  $S \subseteq T \subseteq N$ , and *strictly positive*, i.e.,  $v(T) > 0$  for all  $\emptyset \neq T \subseteq N$ . As  $\mathcal{M}^*(N, v) \subseteq \mathcal{I}^*(N, v)$  it suffices to show that  $x \in \mathcal{I}^*(N, v)$  with  $x_l > v(N)$  for some  $l \in N$  implies that  $x$  is not a member of  $\mathcal{M}^*(N, v)$ . To this end define  $P = \{i \in N \mid x_i < 0\}$  and observe that  $P \neq \emptyset$ , because  $x(N) = v(N) > 0$ . With  $y \in \mathbb{R}^P$  defined by  $y_i = v(P)/|P|$  it suffices to show that  $(P, y)$  constitutes a justified objection of an arbitrary player  $k \in P$  against  $l$ . Indeed, this objection cannot be countered via any coalition  $Q \in \mathcal{T}_{lk}$ , because

$$\begin{aligned} v(Q) - y(Q \cap P) - x(Q \setminus P) &\leq v(Q) - x(Q \setminus P) \leq v(Q) - x_l \\ &< v(Q) - v(N) \leq 0 \end{aligned}$$

is true.

**q.e.d.**

## 2. Properties of the semireactive bargaining set

The main aim of this section is to show that the semireactive prebargaining set satisfies the reduced game property and a weak form of the converse reduced game property. Moreover, it is shown that it also satisfies some other well-known axioms.

Some convenient and well-known properties of a solution  $\sigma$  on a set  $\Gamma$  of games are as follows.

- (1)  $\sigma$  is *anonymous* (satisfies **AN**), if for each  $(N, v) \in \Gamma$  and each bijective mapping  $\tau : N \rightarrow N'$  with  $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

- holds (where  $(\tau v)(T) = v(\tau^{-1}(T))$ ,  $\tau_j(x) = x_{\tau^{-1}j}$  ( $x \in \mathbb{R}^N$ ,  $j \in N'$ ,  $T \subseteq N'$ )).  
 (2)  $\sigma$  is *covariant under strategic equivalence* (satisfies **COV**), if for  $(N, v)$ ,  $(N, w) \in \Gamma$  with  $w = \alpha v + \beta$  for some  $\alpha > 0$ ,  $\beta \in \mathbb{R}^N$

$$\sigma(N, w) = \alpha \sigma(N, v) + \beta$$

holds. The games  $v$  and  $w$  are called *strategically equivalent*.

- (3)  $\sigma$  satisfies *nonemptiness* (**NE**), if  $\sigma(N, v) \neq \emptyset$  for  $(N, v) \in \Gamma$ .  
 (4)  $\sigma$  is *Pareto optimal* (satisfies **PO**), if  $\sigma(N, v) \subseteq \mathcal{S}^*(N, v)$  for  $(N, v) \in \Gamma$ .  
 (5)  $\sigma$  satisfies the *nullplayer property* (**NPP**), if for every  $(N, v) \in \Gamma$  every  $x \in \sigma(N, v)$  satisfies  $x_i = 0$  for every nullplayer  $i \in N$ . Here  $i$  is *nullplayer* if  $v(S \cup \{i\}) = v(S)$  for  $S \subseteq N$ .  
 (6)  $\sigma$  is *reasonable* (satisfies **REAS**), if

$$x_i \geq \min\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} \quad (2.1)$$

and

$$x_i \leq \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} \quad (2.2)$$

for  $i \in N$ ,  $(N, v) \in \Gamma$ , and  $x \in \sigma(N, v)$ .

It is well-known that both, the classical and the reactive (pre)bargaining set satisfy anonymity, covariance under strategic equivalence, and Pareto optimality. Every of the mentioned prebargaining sets also satisfies nonemptiness. Moreover, if  $\Gamma \subseteq \Gamma_{\mathcal{U}}^{\mathcal{F}}$ , then the bargaining sets satisfy NE. The semireactive (pre)bargaining set satisfies PO by definition. A proof of AN and COV is straightforward and left to the reader.

In general none of the mentioned bargaining sets satisfy the nullplayer property or reasonableness (Note that REAS implies NPP). However, in Section 4 it will be shown that the semireactive (pre)bargaining set satisfies REAS on the set of superadditive games.

We recall the definitions of the reduced game (see Davis and Maschler (1965)) and of the reduced game property and its converse (see Sobolev (1975) and Peleg (1986)).

Let  $(N, v)$  be a game, let  $\emptyset \neq S \subseteq N$ , and  $x \in X(N, v)$ . The *reduced game* w.r.t.  $S$  and  $x$  is the game  $(S, v^{S,x})$  defined by

$$v^{S,x}(T) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(N) - x(N \setminus S), & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\}, & \text{otherwise} \end{cases}$$

**Definition 2.1.** Let  $\sigma$  be a solution on a set  $\Gamma$  of games. Then  $\sigma$  satisfies the

- (1) **reduced game property (RGP)**, if the following condition holds: If  $(N, v) \in \Gamma$ ,  $\emptyset \neq S \subseteq N$ , and  $x \in \sigma(N, v)$ , then  $(S, v^{S,x}) \in \Gamma$  and  $x_S \in \sigma(S, v^{S,x})$ .
- (2) **converse reduced game property (CRGP)**, if the following condition holds: If  $(N, v) \in \Gamma$  with  $|N| \geq 3$ , if  $x \in \mathcal{J}^*(N, v)$ , and if for every  $S \subseteq N$  with  $2 \leq |S| \leq |N| - 1$  the reduced game  $(S, v^{S,x})$  is a member of  $\Gamma$  and  $x_S \in \sigma(S, v^{S,x})$ , then  $x \in \sigma(N, v)$ .

Note that the converse reduced game property as defined by Peleg (1986) only requires that the reduced games w.r.t. two-person coalitions have to be taken into consideration. However, for the solutions that have been axiomatized with the help of the converse reduced game property, i.e. the prekernel, the core (Peleg (1986, 1989)), and the positive prekernel (Sudhölter and Peleg (2000)), even our weaker version of the converse reduced game property is suitable to replace the “classical” version in all characterizations.

It is well-known (see Peleg (1988) and Granot and Maschler (1997)) that both the classical prebargaining set and the reactive prebargaining set satisfy the reduced game property. We shall now show that the semireactive prebargaining set satisfies RGP.

**Lemma 2.2.** *The semireactive prebargaining set satisfies the reduced game property.*

*Proof:* Let  $(N, v)$  be a game,  $\emptyset \neq S \subseteq N$ , and  $x \in \mathcal{M}_{sr}^*(N, v)$ . With  $w = v^{S,x}$  we have to show that  $x_S \in \mathcal{M}_{sr}^*(S, w)$ . To this end let  $k, l \in S, k \neq l$  and  $P \in \mathcal{T}_{kl}(S)$  satisfy  $e(P, x_S, w) > 0$ . We have to prove that there exists a coalition  $Q \in \mathcal{T}_{lk}(S)$  such that either  $Q \cap P = \emptyset$  and  $e(Q, x_S, w) \geq 0$  or  $e(Q, x_S, w) \geq e(P, x_S, w)$  (see equation (1.4)). Let  $R \subseteq N \setminus S$  such that  $w(P)$  is attained by  $P \cup R$ , i.e.,  $w(P) = v(P \cup R) - x(R)$ . Then there is a coalition of the form  $Q \cup T \in \mathcal{T}_{lk}(N)$ ,  $Q \subseteq S, T \subseteq N \setminus S$  such that either  $Q \cup T$  does not intersect  $P \cup R$  and possesses a nonnegative excess or the coalitions intersect and  $e(Q \cup T, x, v) \geq e(P \cup R, x, v)$ . In both cases (1.4) is valid. **q.e.d.**

Note that none of the prebargaining sets satisfies CRGP in general. In order to give an example we assume that  $N = \{1, 2, 3\} \subseteq U$  and  $(N, v)$  is given by  $v(\{1, 2\}) = 1, v(\{1, 3\}) = v(\{2, 3\}) = v(\emptyset) = v(N) = 0$ , and  $v(S) = -1$ , otherwise. Then the imputation  $x = (0, 0, 0)$  does not even belong to the classical bargaining set, because  $(\{1, 2\}, (1/2, 1/2))$  is a justified objection of 1 against 3 at  $x$ . However, it can easily be checked that  $x_S \in \mathcal{M}_r^*(S, v^{S,x})$  holds true for every nonempty proper subcoalition  $S$  of  $N$ .

The preceding example motivates the definition of a more restricted version of the converse reduced game property. Indeed, it is possible to weaken the converse reduced game property in such a way that the semireactive prebargaining set satisfies the weaker version. On the other hand our modification of the converse reduced game property together with other properties will be used to characterize the semireactive prebargaining set (see Section 3) by implying that it is the maximum solution that satisfies the remaining axioms. Therefore we shall, on the one side, weaken CRGP by requiring that the “reference” vector  $x$  of Definition 2.1 (2) does not only satisfy Pareto optimality but also an additional condition which we shall call *subgrand stability*.

On the other hand our converse reduced game property will be stronger, because it can even be applied to one-person reduced games of a two-person game. In order to explicitly formulate the suitable converse reduced game property a new axiom is defined.

**Definition 2.3.** A feasible payoff vector  $x \in X(N, v)$  of a game  $(N, v)$  is **subgrand stable** if, for all  $l \in N$  with  $x_l > v(\{l\})$  and  $x(N \setminus \{l\}) < v(N \setminus \{l\})$ , the intersection of all coalitions  $Q$  with  $l \in Q$  and  $e(Q, x, v) \geq e(N \setminus \{l\}, x, v)$  consists of player  $l$  only. A solution  $\sigma$  on a set  $\Gamma$  of games satisfies **subgrand stability (SGS)**, if for all games  $(N, v) \in \Gamma$  all members  $x \in \sigma(N, v)$  are subgrand stable.

In order to give an interpretation of subgrand stability, assume  $l$  has the properties required in Definition 2.3. Then  $N \setminus \{l\}$  is treated unsatisfactorily by  $x$ , because this coalition has a positive excess. Nevertheless the vector  $x$  may be considered a “stable” proposal, because the remaining player  $l$  as well – though treated satisfactorily as a single player – “has” a coalition which contains himself, which does not contain an arbitrary player of  $N \setminus \{l\}$ , and which is treated at least as unsatisfactorily as  $N \setminus \{l\}$ . In the foregoing sense player  $l$  is a very significant player in many coalitions that are treated at least as bad as  $N \setminus \{l\}$ . Indeed, he possesses the following “strong argument” to defend his payoff: “Yes, I know, you altogether have a positive excess but look at all the coalitions containing myself that have an excess that is at least as high as yours. I am the unique member of all of these coalitions. Therefore it is unreasonable to decrease my payoff. Moreover, if you will do that nevertheless, then you will also hurt coalitions of which some of you are members of.”

**Lemma 2.4.** *The semireactive (pre)bargaining set satisfies subgrand stability.*

*Proof:* Let  $(N, v), x, l$  satisfy the conditions of Definition 2.3. If there is some player  $k \neq l$  who belongs to every coalition  $Q$  satisfying  $l \in Q$  and  $e(Q, x, v) \geq e(N \setminus \{l\}, x, v)$ , then  $k$  has a justified objection against  $l$  via  $N \setminus \{l\}$ . Indeed, if  $Q \in \mathcal{T}_{lk}(N)$  satisfies  $e(Q, x, v) \geq 0$ , then  $Q \neq \{l\}$ . However, there is no such  $Q$  which satisfies  $e(Q, x, v) \geq e(N \setminus \{l\}, x, v)$ , thus there is an objection of  $k$  against  $l$  via  $N \setminus \{l\}$  which cannot be countered via  $Q$ . **q.e.d.**

Note that subgrand stability rules out justified objections in the sense of the semireactive bargaining set via any “subgrand coalition”, i.e., a coalition of the form  $N \setminus \{l\}$ . Note furthermore that the reactive (pre)bargaining set satisfies SGS, because it is contained in the semireactive (pre)bargaining set.

With the help of subgrand stability the converse reduced game property can be modified in a suitable way.

**Definition 2.5.** A solution  $\sigma$  on a set  $\Gamma$  of games satisfies the **converse reduced game property restricted to subgrand stable preimputations (CRGP<sub>sr</sub>)**, if the following condition holds: If  $(N, v) \in \Gamma$  with  $|N| \geq 2$ , if  $x \in \mathcal{S}^*(N, v)$  satisfies subgrand stability, and if for every  $S \subseteq N$  with  $1 \leq |S| \leq |N| - 1$  the reduced game  $(S, v^{S, x})$  is a member of  $\Gamma$  and  $x_S \in \sigma(S, v^{S, x})$ , then  $x \in \sigma(N, v)$ .

**Lemma 2.6.** *The semireactive prebargaining set satisfies CRGP<sub>sr</sub> on any set  $\Gamma$  of games.*

*Proof:* Let  $x$  be any subgrand stable preimputation of a game  $(N, v) \in \Gamma$  with at least two players such that the  $(|N| - 1)$ -person reduced games are members of  $\Gamma$ . Moreover, we assume that  $x$  does **not** belong to the semireactive prebargaining set of  $(N, v)$ . By (1.4) there are distinct players  $k$  and  $l$  of  $N$  and a coalition  $P \in \mathcal{T}_{kl}(N)$  with  $e(P, x, v) > 0$  such that for all  $Q \in \mathcal{T}_{lk}(N)$

$$e(Q, x, v) < \begin{cases} 0, & \text{if } Q \cap P = \emptyset \\ e(P, x, v), & \text{if } Q \cap P \neq \emptyset \end{cases}$$

By subgrand stability  $P \neq N \setminus \{l\}$ , thus there is a player  $j \in N \setminus (P \cup \{l\})$ . Let  $S = N \setminus \{j\}$  and  $w = v^{S,x}$ . The straightforward proof that  $k$  has a justified objection against  $l$  in the sense of the semireactive bargaining set at  $x_S$  w.r.t the reduced game (namely via  $P$ ) is skipped. **q.e.d.**

In Section 3 it will be shown that the reactive prebargaining set does not satisfy  $\text{CRGP}_{\text{sr}}$ . Moreover, it will turn out that the classical prebargaining set satisfies this property.

In order to obtain similar results for the semireactive bargaining set the notion of the ‘‘imputation saving’’ reduced game (see Snijders (1995)) is useful. If  $(N, v)$  is a game,  $\emptyset \neq S \subseteq N$  is a coalition and  $x \in X(N, v)$ , then the *imputation saving reduced game*  $(S, v^{S,x})$  is the game defined by

$$\overline{v^{S,x}}(T) = \begin{cases} v^{S,x}(T), & \text{if } |T| \neq 1 \\ \min\{v^{S,x}(T), x(T)\}, & \text{if } |T| = 1 \end{cases}$$

Note that the imputation saving reduced game w.r.t. the grand coalition  $N$  leaves the game unchanged, if and only if the proposal  $x$  is individually rational. Therefore we shall require individual rationality, whenever imputation saving reduced games occur. These considerations directly lead to the following modifications of the definitions of  $\text{RGP}$  and  $\text{CRGP}_{\text{sr}}$ .

A solution  $\sigma$  on a set  $\Gamma \subseteq \Gamma_U^{\mathcal{J}}$  of games with imputations satisfies the

- (1) **reduced game property w.r.t. imputation saving reduced games ( $\overline{\text{RGP}}$ )**, if the following condition holds: If  $(N, v) \in \Gamma$ ,  $\emptyset \neq S \subseteq N$ , and  $x \in \sigma(N, v)$ , then  $(S, \overline{v^{S,x}}) \in \Gamma$  and  $x_S \in \sigma(S, \overline{v^{S,x}})$ .
- (2) **converse reduced game property w.r.t. imputation saving reduced games restricted to subgrand stable imputations ( $\overline{\text{CRGP}}_{\text{sr}}$ )**, if the following condition holds: If  $(N, v) \in \Gamma$  with  $|N| \geq 2$ , if  $x \in \mathcal{S}(N, v)$  is subgrand stable, and if for every  $S \subseteq N$  with  $1 \leq |S| \leq |N| - 1$  the imputation saving reduced game  $(S, \overline{v^{S,x}})$  is a member of  $\Gamma$  and  $x_S \in \sigma(S, \overline{v^{S,x}})$ , then  $x \in \sigma(N, v)$ .

It is well-known that both, the classical and the reactive bargaining set on  $\Gamma_U^{\mathcal{J}}$  satisfy the reduced game property w.r.t. imputation saving reduced games.

**Lemma 2.7.** *The semireactive bargaining set on  $\Gamma_U^{\mathcal{J}}$  satisfies (a)  $\overline{\text{RGP}}$  and (b)  $\overline{\text{CRGP}}_{\text{sr}}$ .*

*Proof:* Assertion (a) can be proved by literally copying – only the reduced game has to be replaced by the imputation saving reduced game – the proof of Lemma 2.2. The fact that the imputation saving reduced game  $(S, \overline{w})$  may only

differ from the reduced game  $(S, w)$ , where  $(S, w)$  is the game defined in the proof of Lemma 2.6, on one-person coalitions and the worth of a one-person coalition  $\{i\}$  differs in both games, if and only if  $w(\{i\}) > x_i$ , thus  $\bar{w}(\{i\}) = x_i$ , shows that the proof of that lemma can be modified in such a way that it proves (b). **q.e.d.**

In Section 3 it will be shown that the classical bargaining set satisfies and the reactive bargaining set does not satisfy  $\overline{\text{CRGP}}_{\text{sr}}$ .

### 3. An axiomatization of the semireactive bargaining set

This section is devoted to axiomatize the semireactive bargaining set and the semireactive prebargaining set. We start with a characterization of the semireactive prebargaining set.

**Theorem 3.1.** *The semireactive prebargaining set is the unique solution that satisfies NE, PO, SGS, RGP and  $\overline{\text{CRGP}}_{\text{sr}}$ .*

The following lemma is useful to prove Theorem 3.1.

**Lemma 3.2.** *If  $\sigma$  is a solution that satisfies PO, SGS, and RGP, then it is a subsolution of the semireactive prebargaining set.*

*Proof:* Let  $\sigma$  have the required properties. By PO  $\sigma(N, v) \subseteq \mathcal{M}_{\text{sr}}^*(N, v)$  holds true for any one-person game of  $\Gamma_U$ . We proceed by induction on the number  $|N|$  of players and assume that the desired inclusion is already shown for all games with less than  $m$  players for some  $m > 1$ . Let  $(N, v) \in \Gamma_U$  be any  $m$ -person game and  $x \in \sigma(N, v)$ . By RGP of  $\sigma$  the restriction  $x_S$  belongs to  $\sigma(S, v^{S,x})$  for every  $\emptyset \neq S \subseteq N$ . This is true in particular, if  $\emptyset \neq S \subset N$ , thus Lemma 2.6 yields  $x \in \mathcal{M}_{\text{sr}}^*(N, v)$ . **q.e.d.**

*Proof of Theorem 3.1:* The semireactive prebargaining set satisfies PO by definition. It satisfies NE (see Section 1), RGP (by Lemma 2.2), SGS (by Lemma 2.4), and  $\overline{\text{CRGP}}_{\text{sr}}$  (by Lemma 2.6). In order to prove the converse assertion, let  $\sigma$  be a solution that satisfies the required properties. In view of Lemma 3.2 it remains to show that the semireactive prebargaining set is a subsolution of  $\sigma$ . To this end let  $(N, v) \in \Gamma_U$  and  $x \in \mathcal{M}_{\text{sr}}^*(N, v)$ . If  $|N| = 1$ , then  $x \in \sigma(N, v)$  by PO and NE. We proceed by induction on  $|N|$  and assume that the inclusion is already verified for all games with less than  $m$  persons for some  $m > 1$ . If  $|N| = m$ , then  $x_S \in \sigma(S, v^{S,x})$  for all  $\emptyset \neq S \subset N$  by RGP of  $\mathcal{M}_{\text{sr}}^*$  and the inductive hypothesis. Moreover,  $x$  is subgrand stable, because  $\mathcal{M}_{\text{sr}}^*$  satisfies SGS. By  $\overline{\text{CRGP}}_{\text{sr}}$  of  $\sigma$  we conclude  $x \in \sigma(N, v)$ . **q.e.d.**

Note that PO and NE are only needed for one-person games. Indeed, if a solution satisfies RGP and if it is Pareto optimal on one-person games, then it satisfies PO. Therefore it is possible to replace PO and NE by one-person rationality. A solution  $\sigma$  on a set  $\Gamma$  of games satisfies *one-person rationality* (OPR), if it contains a preimputation in the case of a one-person game, i.e.,  $\sigma(N, v) = \{v(N)\}$ , whenever  $|N| = 1$ .

The reactive prebargaining set satisfies SGS (see Lemma 2.4), NE, PO (see Section 1), RGP (see Section 2), and does not coincide with the semireactive prebargaining set, if  $|U| \geq 4$  (see Example 1.2), thus it does not satisfy  $\text{CRGP}_{\text{sr}}$ . In Remark 5.5 it will be shown that the prebargaining set satisfies  $\text{CRGP}_{\text{sr}}$ , thus it does not satisfy SGS by the same example, if  $|U| \geq 4$ .

If  $|U| \geq 2$ , then the following examples show the logical independence of NE, PO, SGS, RGP, and  $\text{CRGP}_{\text{sr}}$ . Indeed, if the universe consists of a single player only, then only NE and PO are needed to show the theorem. The empty solution shows the independence of NE. If Definition 1.1 is changed in such a way that only the requirement of Pareto optimality is replaced by the requirement of feasibility (i.e., the phrase “all preimputations  $x$ ” is replaced by the phrase “all feasible payoff vectors  $x$ ”), then the solution satisfies all properties except PO. The set of preimputations  $\mathcal{S}^*$  satisfies all properties except SGS. The solution which assigns to any game the set of all subgrand stable preimputations satisfies all axioms except RGP. Finally the prekernel satisfies all axioms except  $\text{CRGP}_{\text{sr}}$ .

It should be noted that we needed NE only once in the proof of Theorem 3.1, namely to guarantee that a solution that satisfies the axioms contains the semireactive bargaining set, if the attention is restricted to one-person games. Hence NE can be replaced by “nonemptiness for one-person games”.

There is an analogue of Theorem 3.1 for the semireactive bargaining set. A solution  $\sigma$  on  $\Gamma$  is said to satisfy *individual rationality* (IR), if  $x_i \geq v(\{i\})$  for all  $(N, v) \in \Gamma$ ,  $x \in \sigma(N, v)$ , and all  $i \in N$ .

**Theorem 3.3.** *The semireactive bargaining set is the unique solution on  $\Gamma_U^{\mathcal{J}}$  that satisfies NE, PO, IR, SGS,  $\overline{\text{RGP}}$ , and  $\overline{\text{CRGP}}_{\text{sr}}$ .*

Lemma 3.2 has the following analogue.

**Lemma 3.4.** *If  $\sigma$  is a solution on  $\Gamma_U^{\mathcal{J}}$  that satisfies PO, IR, SGS, and  $\overline{\text{RGP}}$ , then it is a subsolution of the semireactive bargaining set.*

*Proof:* The proof is very similar to the one of Lemma 3.2. IR is additionally needed, because the current version of the converse reduced game property requires both, Pareto optimality and individual rationality. **q.e.d.**

*Proof of Theorem 3.3:* The semireactive bargaining set satisfies PO and IR by definition. It satisfies NE (see Section 1),  $\overline{\text{RGP}}$  and  $\overline{\text{CRGP}}_{\text{sr}}$  (by Lemma 2.7), and SGS (by Lemma 2.4). The proof can be completed similarly to the proof of Theorem 3.1. Of course the reduced game has to be replaced by the imputation saving reduced game. **q.e.d.**

Again NE is only used for one-person games. If the definition of the semireactive bargaining set (see Definition 1.1) is changed in such a way that Pareto optimality is only required for one-person games, then the arising solution satisfies all axioms except PO. This example shows that Pareto optimality is also needed for two-person games. Thus NE and PO cannot be replaced by OPR.

The same reasoning as above shows that the reactive bargaining set satisfies all axioms of Theorem 3.3 except  $\overline{\text{CRGP}}_{\text{sr}}$ , if  $|U| \geq 4$ . In Remark 5.5 it will be shown that the prebargaining set satisfies  $\overline{\text{CRGP}}_{\text{sr}}$ , thus it does not satisfy SGS, if  $|U| \geq 4$ .

Let  $|U| \geq 2$ . The individually rational subsolutions of the solutions that show the independence of all axioms used in Theorem 3.1 also show the independence of all axioms used in Theorem 3.3 except IR. The semireactive prebargaining set satisfies all axioms except IR, if  $|U| \geq 3$ . Indeed, it is well-known that the prekernel may not be individual rational even in the three-person case and even if the game has an imputation. The fact that all mentioned prebargaining sets coincide with each other and with the bargaining sets for two-person games, immediately shows that IR can be dropped as a condition in Theorem 3.3, if  $|U| \leq 2$ .

#### 4. The semireactive bargaining set for superadditive games

In this section all considered games  $(N, v)$  are *superadditive*, i.e.,  $v(S) + v(T) \leq v(S \cup T)$  holds, whenever  $S \cap T = \emptyset$ .

A game  $(N, v)$  is called a *simple game*, if it is monotonic and if  $v(N) = 1$  and  $v(S) \in \{0, 1\} \forall S \subseteq N$ . A *winning* coalition  $S$  satisfies  $v(S) = 1$ .

It is well known (and moreover easy to prove) that a simple game has a nonempty core if and only if the set  $V$  of *veto players* is nonempty. Player  $i \in N$  is a *veto player*, if  $v(S) = 0$  whenever  $i \notin S$ . The core of simple games with veto players consists of all distributions of  $v(N) = 1$  among the veto players, i.e.,  $\mathcal{C}(N, v) = \{x \in \mathcal{I}(N, v) \mid x(V) = 1\}$ .

In what follows we shall show that for *arbitrary* superadditive games the semireactive prebargaining set satisfies IR. Moreover, for superadditive *simple games* the semireactive bargaining set is the union of the core and the kernel. Granot, Granot, and Zhu (1997) (Theorem 7) showed the same statement for the reactive bargaining set.

**Theorem 4.1.** *If  $(N, v)$  is a superadditive game, then  $\mathcal{M}_{\text{sr}}^*(N, v) = \mathcal{M}_{\text{sr}}(N, v)$ . If  $(N, v)$  is a superadditive simple game, then*

$$\mathcal{M}_{\text{sr}}(N, v) = \begin{cases} \mathcal{C}(N, v), & \text{if } \mathcal{C}(N, v) \neq \emptyset \\ \mathcal{K}(N, v), & \text{if } \mathcal{C}(N, v) = \emptyset \end{cases}.$$

*Proof:* In order to show the first assertion, let  $(N, v)$  be an arbitrary superadditive game. If  $x \in \mathcal{I}^*(N, v) \setminus \mathcal{I}(N, v)$ , then there is a player  $k$  satisfying  $x_k < v(\{k\})$ . By superadditivity every coalition of maximal excess contains  $k$ . Let  $P$  be a *maximal* (w.r.t. set inclusion) coalition with maximal excess. This maximal excess is positive, because  $e(\{k\}, x, v) > 0$ . Moreover, by superadditivity and maximality,  $P$  contains all players  $j$  with  $x_j \leq v(\{j\})$ . By Pareto optimality  $P \neq N$ . Take  $l \in N \setminus P$  and  $Q \in \mathcal{T}_{lk}$ . Then  $e(Q, x, v)$  is not maximal and  $Q$  can only be used in a counter objection if  $Q \cap P = \emptyset$  and  $e(Q, x, v) \geq 0$ . This is not possible by maximality of  $P$ . Thus  $k$  has a justified objection via  $P$ .

In order to show the second assertion let  $(N, v)$  be a superadditive simple game. We distinguish the following cases.

- (1)  $\mathcal{C}(N, v) \neq \emptyset$ . Let  $x \in \mathcal{I}(N, v) \setminus \mathcal{C}(N, v)$ . It remains to show that  $x \notin \mathcal{M}_{\text{sr}}(N, v)$ . In view of the fact that  $x$  does not belong to the core of  $(N, v)$ , there is some player  $l \in N \setminus V$  satisfying  $x_l > v(\{l\}) = 0$ , where  $V$  denotes the set of veto players. With  $P = N \setminus \{l\}$  we come up with  $e(P, x, v) = 1 - x(P) = 1 - x(N) + x_l = x_l$ . Moreover,  $e(Q, x, v) = -x(Q) \leq -x_l < 0$

holds true for any  $Q \subseteq N$  satisfying  $l \in Q$  and  $V \setminus Q \neq \emptyset$ . These observations directly show that every player in  $V$  has a justified objection against  $l$  via  $P$ .

- (2)  $\mathcal{C}(N, v) = \emptyset$ . Let  $x \in \mathcal{I}(N, v) \setminus \mathcal{K}(N, v)$ . It remains to show that  $x \notin \mathcal{M}_{\text{sr}}(N, v)$ . Indeed, there are distinct players  $k$  and  $l$  such that  $s_{kl}(x, v) > s_{lk}(x, v)$  and  $x_l > v(\{l\})$ . By the absence of veto players we have  $e(N \setminus \{k\}, x, v) = x_k \geq 0$ , thus  $s_{kl}(x, v) > s_{lk}(x, v) \geq 0$ . Let  $P \in \mathcal{T}_{kl}$  be a maximal coalition with  $e(P, x, v) = s_{kl}(x, v)$ . For every coalition  $Q \in \mathcal{T}_{lk}$  we have  $e(Q, x, v) < e(P, x, v)$  and, therefore,  $Q$  can only be used in a counter objection if  $Q \cap P = \emptyset$  and  $e(Q, x, v) \geq 0$ . Then  $Q$  must be a winning coalition, because of  $x_l > 0$ . However, disjoint winning coalitions do not exist in a superadditive simple game. We conclude that  $k$  has a justified objection against  $l$  via  $P$ . **q.e.d.**

*Remark 4.2:* Theorem 4.1 shows that the positive (pre)kernel of a superadditive simple game coincides with its (semi)reactive (pre)bargaining set and with the union of the kernel and the core.

Examples of simple superadditive games are *apex games* (an apex game has a distinguished “strong” player such that a coalition is winning if it contains this strong player and at least one additional “weak” player or if it contains all weak players), *superadditive weighted majority games* in general (a superadditive weighted majority game  $(N, v)$  has a *representation*  $(\lambda, m)$  satisfying  $1/2 < \lambda \leq 1$ ,  $m \in \mathbb{R}_+^N$ ,  $m(N) = 1$ , and  $v(S) = 1$ , iff  $m(S) \geq \lambda$ ), and the *seven person projective game* (see von Neumann and Morgenstern (1953) or Granot and Maschler (1997)). For the seven person projective game as well as for apex games with more than two players the core is empty, thus the semireactive prebargaining set coincides with the kernel in these cases.

As we have seen in Remark 4.2, the semireactive (pre)bargaining set, unlike the classical (pre)bargaining set, satisfies REAS and, thus NPP, on superadditive simple games. The following result shows that these properties hold even on superadditive nonsimple games.

**Theorem 4.3.** *For superadditive games the semireactive prebargaining set is reasonable and satisfies the nullplayer property.*

*Proof:* Let  $(N, v)$  be any superadditive game and  $x \in \mathcal{I}^*(N, v)$ . In view of Theorem 4.1 it remains to show that

$$x_l > \max_{S \subseteq N \setminus \{l\}} v(S \cup \{l\}) - v(S) \quad \text{for some } l \text{ implies } x \notin \mathcal{M}^*(N, v).$$

Note that a coalition of highest excess does not contain player  $l$ . Let  $P \subseteq N$  be maximal coalition of highest excess. In view of the fact that  $e(N \setminus \{l\}, x, v) > e(N, x, v) = 0$  we have  $e(P, x, v) > 0$ . Take  $k \in P$ . Then  $P \in \mathcal{T}_{kl}$ . An objection against  $l$  via  $P$  can not be countered via  $Q \in \mathcal{T}_{lk}$ , because such a coalition  $Q$  does not possess the highest excess and  $P \cap Q = \emptyset$  and  $e(Q, x, v) \geq 0$  is not possible by the maximality of  $P$  and superadditivity. Thus  $k$  has a justified objection against  $l$  via  $P$ . **q.e.d.**

The following example shows that the second assertion of Theorem 4.1 cannot be generalized to superadditive nonsimple games.

*Example 4.4.* Let  $N = \{1, 2, 3, 4, 5\}$  and  $(N, v)$  be given by

$$v(S) = \begin{cases} 1, & \text{if } S \in \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 3, 4\}, \\ & \{2, 3, 5\}, \{1, 2\}, \{1, 5\}\} \\ 0, & \text{if } S \in \{N, \emptyset, \{1, 2, 3, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}, \{4, 5\}\}. \\ -4, & \text{if } S \in \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\} \\ -5, & \text{otherwise} \end{cases}$$

It is easy to check that this game is superadditive and  $x = (0, \dots, 0)$  is an imputation. Furthermore,  $s_{ij}(x, v) = 1$  for all pairs  $(i, j) \neq (4, 1)$ . The coalitions in the first row separate each pair of players  $(i, j)$  except  $(4, 1)$  and  $s_{1,4}(x, v) = 1 > s_{4,1}(x, v) = 0$ . So,  $x$  is neither in the core nor in the kernel of the game. It is nevertheless an element of the (semi)reactive bargaining set. We only need to consider objections of player 1 against player 4 and, for this purpose, only the coalitions  $\{1, 2\}$  and  $\{1, 5\}$  can be used. The coalition  $\{3, 4\}$  is disjoint from both coalitions and can counter every objection via  $\{1, 2\}$  or  $\{1, 5\}$ .

The literature provides several classes of superadditive *balanced* games for which the bargaining set and the core coincide. These results immediately apply to the semireactive and to the reactive (pre)bargaining set.

The following classes have been proved (see Solymosi (1999)) to have this property:

- (1) convex games (Maschler, Peleg and Shapley (1972)),
- (2) strongly balanced partitioning games, including, e.g., assignment games and  $\Gamma$ -component additive games (see Potters and Reijnders (1995)),
- (3) simple network games (for these games Granot (1994) and Granot, Granot, and Zhu (1997) showed the coincidence of the core and the reactive bargaining set),
- (4) nonnegative superadditive games  $(N, v)$  with a veto player (i.e. a player  $i$  such that  $v(S) = 0$  if  $i \notin S$ ) (see Potters, Muto and Tijs (1988))

## 5. A characterization of the bargaining set

In this section we show that SGS can be weakened in such a way that, together with the accordingly modified versions of the converse reduced game property, the analogues of Theorems 3.1 and 3.3 can be formulated to characterize the classical (pre)bargaining set. Note that SGS is used to rule out justified objections via subgrand coalitions in the sense of the semireactive bargaining set. A similar property for the bargaining set can be defined with the help of the following notions. If  $(N, v)$  is a game and  $x \in X(N, v)$ , then, for every player  $l \in N$ , we define

$$\mathcal{T}_l(x) = \{Q \subseteq N \mid l \in Q \text{ and } e(Q, x, v) > 0\}.$$

A collection  $\mathcal{Q} \subseteq \mathcal{T}_l(x)$  is *less satisfied than*  $N \setminus \{l\}$ , if there is a mapping  $\lambda_Q \rightarrow \mathbb{R}$ ,  $Q \mapsto \lambda_Q > 0$ , such that

- (1)  $\sum_{Q: k \in Q, Q \in \mathcal{Q}} \lambda_Q \leq 1$  for all  $k \neq l$  and
- (2)  $\sum_{Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) \geq e(N \setminus \{l\}, x, v)$ .

For any collection  $\mathcal{Q}$  of coalitions we define the *support* by  $D(\mathcal{Q}) = \bigcup_{Q \in \mathcal{Q}} Q$ . We use the convention that  $N$  is the support of the empty collection.

**Definition 5.1.** A feasible payoff vector  $x \in X(N, v)$  of a game  $(N, v)$  is **subgrand stable in the sense of the bargaining set** (satisfies  $\text{SGS}_1$ ) if, for all  $l \in N$  with  $x_l > v(\{l\})$  and  $x(N \setminus \{l\}) < v(N \setminus \{l\})$ , the intersection of the supports  $D(\mathcal{Q})$  of all collections  $\mathcal{Q} \subseteq \mathcal{F}_1(x)$  less satisfied than  $N \setminus \{l\}$  consists of player  $l$  only. A solution  $\sigma$  on a set  $\Gamma$  of games satisfies  $\text{SGS}_1$ , if for all games  $(N, v) \in \Gamma$  all members  $x \in \sigma(N, v)$  satisfy  $\text{SGS}_1$ .

If  $Q \in \mathcal{F}_1(x)$  satisfies  $e(Q, x, v) \geq e(N \setminus \{l\}, x, v)$ , then, by putting  $\lambda_Q = 1$ , it follows that  $\{Q\}$  is less satisfied than  $N \setminus \{l\}$ . Hence, “SGS” (see Definition 2.3) implies “ $\text{SGS}_1$ ”.

**Lemma 5.2.** *Let  $(N, v)$  be a game and  $x \in \mathcal{F}^*(N, v)$ . Then  $x$  satisfies  $\text{SGS}_1$ , if and only if there are no distinct players  $k$  and  $l$  of  $N$  such that  $k$  possesses a justified objection against  $l$  at  $x$  via  $N \setminus \{l\}$  in the sense of the bargaining set.*

*Proof:* Let  $x \in X(N, v)$  and suppose that a player  $k \in N \setminus \{l\}$  has a justified objection against player  $l$  via  $N \setminus \{l\}$ , i.e., a justified objection of the form  $(N \setminus \{l\}, x_{N \setminus \{l\}} + z)$ . We conclude that  $z$  satisfies  $z \gg 0$  and  $z(N \setminus \{l\}) = e(N \setminus \{l\}, x, v)$ . As  $l$  has no counter objection, we have  $x(Q) + z(Q \setminus \{l\}) > v(Q)$  for all  $Q \in \mathcal{F}_{lk}$ .

We prove that  $k \in D(\mathcal{Q})$  for every collection  $\mathcal{Q} \subseteq \mathcal{F}_1(x)$  that is less satisfied than  $N \setminus \{l\}$ . Indeed, take weights  $(\lambda_Q)_{Q \in \mathcal{Q}}$  as in the definition. Then

$$\begin{aligned} e(N \setminus \{l\}, x, v) &\leq \sum_{Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) \\ &\leq \sum_{Q: k \notin Q, Q \in \mathcal{Q}} \lambda_Q z(Q \setminus \{l\}) + \sum_{Q: k \in Q, Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) \\ &\leq z(N \setminus \{k, l\}) + \sum_{Q: k \in Q, Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) \\ &< e(N \setminus \{l\}, x, v) + \sum_{Q: k \in Q, Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) \end{aligned}$$

and, thus,  $k \in D(\mathcal{Q})$ .

In order to prove the converse implication, suppose that  $x \in X(N, v)$ ,  $x_l > v(\{l\})$  and  $x(N \setminus \{l\}) < v(N \setminus \{l\})$ .

If  $k$  is a member of the supports  $D(\mathcal{Q})$  of all collections  $\mathcal{Q}$  less satisfied than  $N \setminus \{l\}$ , then it suffices to prove that player  $k$  has a justified objection against player  $l$  via coalition  $N \setminus \{l\}$ . In order to show this claim we shall show that there is a vector  $z \in \mathbb{R}^{N \setminus \{l\}}$  with  $z \gg 0$ ,  $z(N \setminus \{l\}) = e(N \setminus \{l\}, x, v)$  and  $x(Q) + z(Q \setminus \{l\}) > v(Q)$  for all coalitions  $Q \in \mathcal{F}_{lk}$ .

With player set  $N' = N \setminus \{k, l\}$  we define the following “excess” game  $(N', u)$  by

$$u(S) = \begin{cases} (e(S \cup \{l\}, x, v))_+, & \text{if } S \neq N' \\ e(N \setminus \{l\}, x, v), & \text{if } S = N' \end{cases}$$

Assume that there exists a core element  $\hat{z}$  of  $(N', u)$  satisfying  $\hat{z}(S) > u(S)$  for all  $\emptyset \neq S \subset N'$ . Then  $e(N \setminus \{l\}, x, v) > e(N \setminus \{k\}, x, v)$ , because  $\{N \setminus \{k\}\}$  is not less satisfied than  $N \setminus \{l\}$ . As  $\hat{z}(N') = e(N \setminus \{l\}, x, v)$ , there exists  $z \in \mathbb{R}^{N \setminus \{l\}}$  with  $z_{N'} \leq \hat{z}$  and  $z(S) > u(S)$  for all  $\emptyset \neq S \subset N'$  such that  $e(N \setminus \{k\}, x, v) < z(N') < \hat{z}(N')$  and  $z(N \setminus \{l\}) = u(N')$ . We conclude that  $z \gg 0$ , because  $z_k = \hat{z}(N') - z(N') = u(N') - z(N') > 0$ . By definition of  $u$ ,  $(N \setminus \{l\}, x_{N \setminus \{l\}} + z)$  is an objection of  $k$  against  $l$  which cannot be countered via any coalition  $S \cup \{l\}$ ,  $S \subset N'$ . Also,  $l$  is not able to counter this objection via  $N \setminus \{k\}$ , because it  $z(N') > e(N \setminus \{k\}, x, v)$ , hence  $(N \setminus \{l\}, x_{N \setminus \{l\}} + z)$  is a justified objection of  $k$  against  $l$ .

Therefore, it remains to prove that the interior of the core of  $(N', u)$  is nonempty. By a slight modification of the Bondareva-Shapley Theorem (see Bondareva (1963) or Shapley (1967)) this is equivalent with

$$\sum_{S \subset N'} \lambda_S u(S) < u(N') \quad \text{whenever } \lambda_S \geq 0 \quad \forall S \subset N', S \neq \emptyset, N' \quad \text{and}$$

$$\sum_{S \subset N'} \lambda_S \mathbf{1}_S = \mathbf{1}_{N'}.$$

(Here  $\mathbf{1}_S$  denotes the indicator function of  $S$  in  $N'$ .) In order to show this assertion let  $(\lambda_S)_{S \subset N'}$  satisfy  $\lambda_S \geq 0$  and  $\sum_{S \subset N'} \lambda_S \mathbf{1}_S = \mathbf{1}_{N'}$ . Then  $\mathcal{Q} = \{S \cup \{l\} \mid S \subset N', \lambda_S > 0, u(S) > 0\} \subseteq \mathcal{F}_l(x)$  by definition. Moreover,  $k \notin D(\mathcal{Q})$ . With  $\lambda_Q := \lambda_S$ , whenever  $Q = S \cup \{l\}$  for any  $S \subset N'$ , we obtain  $\sum_{Q: j \in Q, Q \in \mathcal{Q}} \lambda_Q \leq 1$  for all  $j \in N \setminus \{l\}$ . By our assumption  $\mathcal{Q}$  is **not** less satisfied than  $N \setminus \{l\}$ , thus the observation

$$\sum_{S \subset N'} \lambda_S u(S) = \sum_{Q \in \mathcal{Q}} \lambda_Q u(Q \setminus \{l\}) = \sum_{Q \in \mathcal{Q}} \lambda_Q e(Q, x, v) < e(N \setminus \{l\}, x, v) = u(N')$$

finishes the proof. **q.e.d.**

**Corollary 5.3.** *The (pre)bargaining set satisfies  $\text{SGS}_1$ .*

The new version of subgrand stability is used to define the suitable version of the converse reduced game property. Indeed,  $\text{CRGP}_1$  is defined as  $\text{CRGP}_{\text{sr}}$  (see Definition 2.5), except that the expression “if  $x \in \mathcal{F}^*(N, v)$  satisfies subgrand stability” is replaced by “if  $x \in \mathcal{F}^*(N, v)$  satisfies  $\text{SGS}_1$ ”. Moreover,  $\overline{\text{CRGP}}_1$  is the property which arises from  $\text{CRGP}_1$  by replacing the reduced game by the imputation saving reduced game.

**Lemma 5.4.** *The prebargaining set satisfies  $\text{CRGP}_1$  and the bargaining set satisfies  $\overline{\text{CRGP}}_1$ .*

*Proof:* In order to show that the (pre)bargaining set satisfies the asserted version of the converse reduced game property, let  $(N, v)$  be a game and  $x \in \mathcal{F}^*(N, v)$  satisfy  $\text{SGS}_1$ . If player  $k$  has a justified objection  $(P, y)$  against some other player  $l$  at  $x$  in the sense of the bargaining set, then  $P \neq N \setminus \{l\}$  by Lemma 5.2. Take  $j \in N \setminus (P \cup \{l\})$ , denote by  $S = N \setminus \{j\}$  the set of remaining players, and let  $w = v^{S, x}$  denote the coalitional function of the reduced game. The fact that  $e(\{j, l\}, x, v) < 0$  directly implies  $w(\{l\}) < x_l$ , thus  $(P, y)$  is a justified objection against  $l$  even w.r.t. the reduced game. The imputation saving reduced game  $(S, \overline{v^{S, x}})$  can be treated in the same way. **q.e.d.**

*Remark 5.5:* A solution  $\sigma$  that satisfies  $\text{CRGP}_1$  or  $\overline{\text{CRGP}}_1$  also satisfies  $\text{CRGP}_{\text{sr}}$  or  $\overline{\text{CRGP}}_{\text{sr}}$ , respectively, because  $\text{SGS}$  implies  $\text{SGS}_1$ . Hence, by Lemma 5.4,  $\mathcal{M}^*$  and  $\mathcal{M}$  satisfy  $\text{CRGP}_{\text{sr}}$  and  $\overline{\text{CRGP}}_{\text{sr}}$ , respectively.

**Theorem 5.6.** (1) *The prebargaining set  $\mathcal{M}^*$  is the unique solution that satisfies NE, PO,  $\text{SGS}_1$ , RGP and  $\text{CRGP}_1$ .*

(2) *The bargaining set  $\mathcal{M}$  is the unique solution on  $\Gamma_U^{\mathcal{F}}$  that satisfies NE, PO, IR,  $\text{SGS}_1$ ,  $\overline{\text{RGP}}$ , and  $\overline{\text{CRGP}}_1$ .*

*Proof:* Both solutions satisfy  $\text{SGS}_1$  by Lemma 5.2. Lemma 5.4 shows that they satisfy the asserted versions of the converse reduced game property. It is well-known that the remaining properties are satisfied.

Uniqueness can be proved as uniqueness was shown in the proofs of Theorem 3.1 and Theorem 3.3. **q.e.d.**

Note that OPR can be used to replace PO and NE in assertion (1) of Theorem 5.6. Suitable modifications of the examples presented in Section 3 show the logical independence of the axioms in both assertions.

## 6. Concluding remarks

*Remark 1:* It is possible to extend the definition of the semireactive prebargaining set to TU games with coalition structures. In order to do so the set of (pre)-imputations (with respect to the grand coalition) has to be replaced by the set of (pre)imputations with respect to the coalition structure (see, e.g., Granot and Maschler (1997)). For simplicity reasons we only considered the case in which the coalition structure is trivial, i.e., consists of the grand coalition.

*Remark 2:* There is a set-valued dynamic system leading to the semireactive (pre)bargaining set. In view of the fact that Section 8 of Granot and Maschler (1997) can suitably be modified in order to generate analogous results for the semireactive bargaining set, we only present a very brief description of the dynamic system. Let  $(N, v)$  be a TU game and  $x \in \mathcal{I}^*(N, v)$  be a preimputation. For coalitions  $P, Q \subseteq N$  define

$$d^{P,Q}(x) = \begin{cases} (\min\{e(P, x, v), -e(Q, x, v)\})_+, & \text{if } P \cap Q = \emptyset \\ (1/2 \cdot (e(P, x, v) - e(Q, x, v)))_+, & \text{otherwise} \end{cases}$$

and for  $k, l \in N$  with  $k \neq l$  define

$$d_{kl}(x) = \max_{P \in \mathcal{F}_{kl}} \min_{Q \in \mathcal{F}_{lk}} d^{P,Q}(x).$$

Moreover,  $y \in \mathbb{R}^N$  is said to arise from  $x$  by a  $d$ -bounded transfer (from  $l$  to  $k$ ), if there exists  $0 \leq \alpha \leq d_{kl}(x)$  such that

$$y_i = \begin{cases} x_k + \alpha, & \text{if } i = k \\ x_l - \alpha, & \text{if } i = l \\ x_i, & \text{otherwise} \end{cases}.$$

The dynamic system  $\varphi$  is the correspondence on  $\mathcal{I}^*(N, v)$  defined by

$$\varphi(x) = \{y \in \mathbb{R}^N \mid y \text{ arises from } x \text{ by a } d\text{-bounded transfer}\}.$$

By (1.4) the semireactive prebargaining set coincides with the set of *endpoints*, i.e.,

$$\mathcal{M}_{\text{sr}}^*(N, v) = \{x \in \mathcal{I}^*(N, v) \mid \varphi(x) = \{x\}\}.$$

A *trajectory* is a sequence  $(x^m)_{m \in \mathbb{N}}$  such that  $x^{m+1}$  arises from  $x^m$  by a  $d$ -bounded transfer. The trajectory is *maximal*, if infinitely often the size of the transfer from  $x^m$  to  $x^{m+1}$  is at least  $\delta \cdot \max_{k,l \in N, k \neq l} d_{kl}(x^m)$  for some  $\delta > 0$ . Applying some results of Maschler and Peleg (1976) (see also Stearns (1968)) we obtain the following assertions. Every maximal trajectory converges to an element of the semireactive prebargaining set. Moreover, if the attention is restricted to imputations only (and if there are imputations), then every maximal trajectory converges to an element of the semireactive bargaining set.

*Remark 3:* It should be noted that the definition of the semireactive (pre)-bargaining set can be extended to cooperative games without transferable utility. Indeed, if the notion of objections and counter objections is taken from, e.g., Asscher (1976), then it is obvious how to generalize Definition 1.1.

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