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Airport problems and consistent allocation rules

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Abstract

A class of single valued rules for airport problems is considered. The common properties of these rules are efficiency, reasonableness and a weak form of consistency. These solutions are automatically members of the core for the associated airport game. Every weighted Shapley value, the nucleolus, and the modified nucleolus turn out to belong to this class of rules. The τ-value, however, does not to belong to this class. As a side result we prove that, for airport games, the modified nucleolus and the prenucleolus of the dual game coincide. Furthermore, we investigate monotonicity properties of the rules and axiomatize the Shapley value, nucleolus, and modified nucleolus on the class of airport games.

Keywords: Airport problem; Airport game; Weighted Shapley value; Nucleolus; Modified nucleolus

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1. Introduction

Airport games were introduced by Littlechild and Owen (1973). The problem they pose is how the cost of a landing strip should be distributed among users who need runways of different lengths. Further discussion of the model can be found e.g. in Littlechild and Thompson (1977) and Dubey (1982). These authors studied the Shapley value and the nucleolus for this kind of cost sharing problems.

In an airport problem there is a finite population N and a nonnegative cost function C: $N \rightarrow \mathbf{R}_+$. For technical reasons it is assumed that the population is taken from the set

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of the natural numbers: players are identified with their ‘ranking number’. The cost function satisfies the inequality $C(i) \leq C(j)$ whenever $i \leq j$. It is typical for airport problems that the cost $C(i)$ is assumed to be a part of the cost $C(j)$ if $i < j$, i.e. a coalition S is confronted with costs $c(S) := \max_{i \in S} C(i)$. In this way an airport problem generates an *airport game* (N, c). As the value of each one-person coalition (i) equals $C(i)$, we can rediscover the airport problem from the airport game. In the paper we investigate rules for airport problems (games) and the properties they have. In this introduction we give a short verbal description of and a motivation for the properties we shall meet in the subsequent sections. Many of these properties have a wider scope; they can be defined for rules on the set of all TU-games as well.

The easiest property is *efficiency*, saying that the contributions of the players cover exactly the total costs. This is such an obvious requirement that one could even be inclined to consider only efficient rules as ‘real rules’: this is what we want to achieve.

In the theory of TU-games a rule is called *reasonable* if the payment of each player is between his smallest and his largest marginal. As the cost games related to airport games turn out to be concave, the largest marginal is $c(i)$, the *stand-alone costs* as Moulin and Shenker (1992) called it, and the smallest marginal is $c(N) - c(N \backslash i)$. Moreover, the latter marginal is often zero. Only if i is a player with the largest needs and there are no other players with the same needs, the smallest marginal is positive. The *reasonableness property* for airport problems says that each player is paying an amount between 0 and his stand-alone costs. He is not paying more than ‘what he uses’ and his co-players are not subsidizing him.

Covariance is also a property adopted from the theory of the TU-games. It says that the addition of an additive game changes the solution accordingly. For airport games the addition of an additive game brings us mostly outside the class of airport games. So, the scope of this axiom is very limited: only the addition of games generated by vectors of the form $(0, 0, 0, \ldots, x)$ with $x \geq 0$ can be taken into consideration. It also says that multiplication of the cost function with a positive factor yields a solution that is multiplied with the same factor.

An important role is played by *consistency* properties. If we have an airport problem and a player i has agreed to pay the contribution x – the amount he has to pay according some rule – the remaining players are left with the same airport problem but one player less and an amount x that can be used to cover a part of the total costs. The question which part of the costs is covered allows for different answers and each of them leads to a different kind of *reduced airport problem*. As it is, however, player i’s intention to pay only for the part he will use, it seems natural that his payment should be subtracted from the costs of this part. So, every player with higher needs (than i) benefits completely from this reduction: $C(j) \to C(j) - x$, whenever $j > i$. For the remaining players k (with $k < j$) it is important to know where the payment x of player i is used for. There are two simple (and many complicated) answers to that question. In the terminology of landing strips, one can start to cover the costs $C(i)$ from the starting point or from the end point of the landing strip needed by i. The first possibility leads to a new cost function $(C(k) - x)_+$ for $k < i$ and the latter possibility to a new cost function $\min\{C(k), C(i) - x\}$ for $k < i$. We call these two kinds of reduced problems the *ψ-reduced* and the *ν-reduced airport problem*. Of course, some technical conditions must be satisfied to make sure that the reduced problem is an airport problem again. However, by assuming reasonableness these conditions are satisfied a great deal. If the leaving player i is the player with the lowest needs, both kinds of reduction agree and, in fact, every reduction that subtracts the amount x from the costs of the part used by the first player gives the same reduced airport problem. In general, consistency says that the rule prescribes for every player in the reduced problem the same contribution as in the original problem. So we get here two consistency concepts called *ψ-consistency* and *ν-consistency*. The letters ψ and ν refer to the notation for the modified nucleolus and the nucleolus that is often used. As will be proved in Section 5 *ψ-consistency* and *ν-consistency* are the most characteristic properties for the modified nucleolus (ψ) and the nucleolus (ν), respectively.

If we require the consistency property only for a player with the lowest needs, only one kind of reduction is possible and we call the consistency property *first-player consistency*. It is a property shared by many rules. If a rule satisfies first-player consistency, the rule is completely determined by the payment of the 'first' player in each airport problem, since the payment of the 'second' player is the payment of the 'first' player in the *reduced airport problem* and so on. The function assigning to each airport problem the payment of the 'first' player is called the *generating function of the rule*. To deal with this concept it must be possible to identify in each population the 'first',the 'second' up to the 'last' player. For that reason we assumed that the finite population in an airport problem was always taken from (a subset of) the natural numbers. Players are identified by their ranking number (like athletes at the Olympic Games).

To summarize, many ways to define reduced airport problems are possible. In this paper we investigate two of them leading to ψ- and ν-consistency. If the leaving player is a player with lowest needs (the first player in the population), all kinds of reduction gives the same reduced problem, when we do justice to the intention of the leaving player to pay for his own part only. If a rule is not completely ad-hoc, one may expect that it exhibits some kind of consistency. The problem, however, is to find the kind of reduction that is needed. The answer to that question gives an idea about the 'nature of the rule'. Therefore, it is not the consistency property that needs a motivation but the way in which the reduced airport problem is defined. The ψ- and the ν-reduction are quite natural ways to come to such a reduction. A different class of desirable properties are the monotonicity properties. It stipulates that players with higher costs are also paying a (weakly) larger contribution. We follow Moulin and Shenker (1992) again in calling this property *fair ranking*. Next we investigate what happens with the solution if the cost of exactly one player increases or if the size of the population (player set) increases. It hardly needs saying that it would be nice if the player with the increased cost would be going to pay more (*cost monotonicity*) or if every member of the population was benefitting from an increase of the population size (*population monotonicity*), certainly if the total costs remain the same.

In this paper we define and investigate cost allocation rules for airport problems. In Section 2 we recall the definition of an airport problem and introduce the idea of first-player consistency and the thereby defined generating function. We show that, under mild conditions, a rule satisfying first-player consistency is a 'core selector'.

Section 3 presents known rules like the weighted Shapley value (Kalai and Samet, 1988), the nucleolus (Schmeidler, 1969), the modified nucleolus (Sudhölter, 1996, 1997) and the prenucleolus of the dual game as first-player consistent rules. In fact, the modified nucleolus and the prenucleolus of the dual cost game coincide on the class of airport games. The τ-value is *not* first-player consistent on the class of airport games.

In Section 4 we investigate monotonicity properties of these solution concepts. The properties we will discuss are strong monotonicity (Young, 1985), coalitional monotonicity (Young,1985), population monotonicity (Thomson, 1993, Sprumont, 1990) and fair ranking (Moulin and Shenker, 1992). Fair ranking and population monotonicity are common properties of the Shapley value, the nucleolus, and the modified nucleolus. The weighted Shapley values are known to be strongly monotonic (Young, 1985). The nucleolus-like solution concepts turn out to be coalitionally monotonic but not strongly monotonic.

In the last section we axiomatize the Shapley value, the nucleolus, and the modified nucleolus on the class of airport games using common axioms like efficiency, the equal treatment property and covariance. We also use specific axioms like strong monotonicity, ν-consistency and ψ-consistency, respectively.

2. Definition of airport problems and airport games

Let us assume that in a community a list of public projects is under discussion and that the projects can be ordered $p_1 < p_2 < \dots < p_q$ where $p_i < p_j$ means that project p_j is an extension of project p_i. The members of the community have given their opinion about the different plans and n_i members voted for project p_i. The coalition of people voting for project p_i is denoted by N_i. We assume that $n_i \geq 1$ for $i = 1, \dots, q$. The realization of project p_i generates cost equal to $C(i)$ and we assume that $C(i) < C(j)$ if $p_i < p_j$ and that $C(i) \geq 0$ for every project p_i. The community authorities decide to realize the largest project p_q and to charge the community members with the cost $C(q)$ to be made.

The paper studies cost allocation rules for this type of problems, rules that take the voting behavior of the community members into consideration.

We call such problems *airport (cost sharing) problems*, just as in the literature, where the projects p_i are landing strips of different lengths.

To be able to deal with non-symmetric solutions and to have an environment in which ‘reduced airport problems’ make sense, we choose the following framework. Let $U \subseteqq IN$ be a universe of players. An *airport problem* consists of a finite nonvoid player set $N \subseteqq U$ and a map $C: N \to IR_+$ satisfying

$$C(i) \leq C(j), \textit{ if } i < j \text{ and } i, j \in N.$$

Sometimes we will understand C as a nonnegative vector in IR_+^N. For every coalition $S \subseteqq N$ we will use

$$l^S = \max S, \text{ and } s = |S|$$

to denote the last player in S and the number of players in S, respectively. The members of N ordered after an increasing index will also be denoted by $[1], \dots, [n]$ where $n = |N|$. So, e.g., $[n] = l^N$. To avoid an unwieldy notation we delete brackets if there is no danger for confusion. We write $C[i]$, $N \setminus [i]$ and $c[i]$ instead of $C([i])$, $N \setminus \{[i]\}$ and $c(\{[i]\})$.

The set of airport problems on U will be denoted by $\mathscr{A}$.

An *(allocation) rule* σ (on $\mathscr{A}$) assigns a payment vector $\sigma(N, C) \in IR^N$ to each airport problem (N, C) in $\mathscr{A}$. Also $\sigma(N, C)$ will sometimes be understood as a vector in IR^N. So we will deal with *single valued* solutions all the time.

A rule σ is called *efficient* if

$$\sum_{i \in N} \sigma_i(N, C) = \max_{i \in N} C(i) = C(l^N) = C[n].$$

Defining efficiency in this way implies that we understand the cost $C(i)$ as a 'part' of the cost $C(j)$ if $C(i) \le C(j)$ (cf. the story we started with).

The next property of rules is the main topic of this paper. Suppose we have a rule σ. Then $x = \sigma_{[1]}(N, C)$ is the payment of the first player of N, when the rule σ is applied. If only this payment has been done, the remaining players $j \in N \setminus [1]$ are faced with a cost function $\bar{C}(j) = C(j) - x$. These considerations motivate the following definition.

Definition 2.1.

1. Let $(N, C) \in \mathscr{A}$ be an airport problem with $n \ge 2$ and $x \in IR$. Define $C^{[1],x}$: $N \setminus [1] \to IR$ by

   $$C^{[1],x}(j) = C(j) - x \textit{ for } j \in N \setminus [1].$$

   If $C^{[1],x} \ge 0$, then $(N \setminus [1], C^{[1],x})$ is the **reduced** airport problem of (N, C) with respect to $[1]$ and x.
2. A rule σ on $\mathscr{A}$ is **first-player consistent,** if for every $(N, C) \in \mathscr{A}$ with $n \ge 2$ the following conditions are satisfied for $x = \sigma(N, C)$:
   2.1. $(\bar{N}, \bar{C}) := (N \setminus [1], C^{[1],x_{[1]}}) \in \mathscr{A}$ and
   2.2. $\sigma(\bar{N}, \bar{C}) = x_{\bar{N}}$.
3. For a first-player consistent rule σ the **associated generating function** $g^\sigma: \mathscr{A} \to IR$ is defined by

   $$g^\sigma(N, C) = \sigma_{[1]}(N, C) \textit{ for } (N, C) \in \mathscr{A}.$$

4. A function $g: \mathscr{A} \to IR$ satisfying

   $$g(N, C) \le C(j) \textit{ for } j \in N \setminus [1]$$

   for $(N, C) \in \mathscr{A}$ with $n \ge 2$, is called a **generating function**.
5. A generating function g **generates** the rule σ^g recursively by
   5.1. $\sigma^g_{[1]}(N, C) = g(N, C)$ and
   5.2. $\sigma^g_{N \setminus [1]}(N, C) = \sigma^g(N \setminus [1], C^{[1], g(N,C)})$, if $n \ge 2$.

Remark 2.2.

1. *Note that first-player consistency implies that the rule σ is completely determined by the payment $\sigma_{[1]}(N,C)$ made by the first player in each problem.*
2. *If σ is a first-player consistent rule, then g^{σ} is a generating function and $\sigma^{g^{\sigma}} = \sigma$.*
3. *If g is a generating function, then σ^{g} is a first-player consistent rule and $g^{\sigma^{g}} = g$.*

A further property of rules is needed. We call a rule σ *reasonable (on both sides)* (cf. Milnor, 1952 and Sudhölter, 1997) if, for all $j \in N$, $0 \le \sigma_j(N, C) \le C(j)$ for all airport problems $(N, C) \in \mathscr{A}$.

Example 2.3. The condition $\sigma_j(N, C) \le C(j)$ is also 'reasonable' in a non-technical meaning of the word. No player pays more than the total cost of the plan (s)he voted for. The other condition $\sigma_j(N, C) \ge 0$ is less 'reasonable' than it seems to be. The following rather convincing rule does not satisfy the second condition: $\rho_j(N, C) = C(j) - \lambda$ where λ is chosen such that the rule is efficient. So every player j pays the total costs of his plan $C(j)$ but gets some 'discount' λ to make the solution efficient. In fact $\lambda = \Sigma_{i \in N \setminus [n]} C(i)/n$. It is easy to check that this rule is first-player consistent, efficient, and satisfies the first reasonability condition. We use this rule as a counter example at the end of the paper. When we discuss the modified nucleolus in the next section, we will meet a variant of this rule.

First-player consistent, efficient and reasonable solutions of airport problems will be the topic of this paper. To say it differently, we will investigate rules generated by functions g: $\mathscr{A} \to IR$ satisfying

$$g(N, C) = C[1], \textit{ if } |N| = 1 \textit{ and } 0 \le g(N, C) \le C[1], \text{ otherwise.}$$

If (N, C) is an airport problem, we define the associated *airport game* as the cooperative cost game (N, c) with coalition values $c(S) = \max_{i \in S} C(i)$. Notice that the values of C can be rediscovered from the game (N, c), because $C(i) = c(i)$. Therefore the airport problem and the associated cost game are frequently identified.

Every airport game is clearly a concave game and has, therefore, a nonempty core. The first theorem states that rules satisfying first-player consistency, efficiency, and reasonableness are core selectors.

Theorem 2.4. *If σ is a rule on $\mathscr{A}$ that satisfies first-player consistency, efficiency and reasonableness, then $\sigma(N, C) \in$ Core (N, c) for every airport problem $(N, C) \in \mathscr{A}$ and the associated cost game (N, c).*

Proof. The proof is by induction on the number n of players. If there is one player, the theorem follows from efficiency. Suppose that the theorem has been proved for problems with less than k players and suppose that (N, C) is an airport problem with k players. Let $x = \sigma(N, C)$. By first-player consistency we have σ $(\bar{N}, \bar{C}) = x_{\bar{N}}$, where $\bar{N} = N\setminus[1]$,

$\bar{C} = C^{[1],x_{[1]}}$. By the inductive hypothesis this is a core allocation of the game $(\bar{N}, \bar{c})$ associated with the reduced airport problem $(\bar{N}, \bar{C})$. If $S \subseteq N$, then

$$\bar{c}(S) = \max_{i \in S} \bar{C}(i) = \max_{i \in S} C(i) - x_{[1]} = c(S) - x_{[1]}.$$

In view of $x \geq 0$, reasonableness, and $x(S) \leq \bar{c}(S)$ we have $x(S) \leq c(S)$. If S is a coalition containing player [1], we have $x(S \setminus [1]) \leq \bar{c}(S \setminus [1]) = c(S \setminus [1]) - x_{[1]}$ and therefore, also in this case, $x(S) \leq c(S)$. □

3. First-player consistency of solution concepts for airport games

In this section we start with the airport game associated with an airport problem and investigate which single valued solution concepts for TU-games are first-player consistent. The solution concepts we will consider are the nucleolus, the prenucleolus of the dual game (also called the antinucleolus of the game), the weighted Shapley value, the modified nucleolus and the τ-value. All these rules will turn out to be first-player consistent except the τ-value. For each of the first-player consistent rules we will give the generating function.

3.1. The weighted Shapley value

We start with the weighted Shapley value. Let us briefly repeat the definition (see Kalai and Samet, 1988 for more details). Like the classical Shapley value, the weighted Shapley value is a linear function of the game. So it is enough to define the solution for the elements of a basis of the vector space G^N of all TU-games with player set N. For this basis the set of *representation games* $\{(N, u_S^*)\}_{S \subseteq N}$, the set of duals of the unanimity games (N, u_S), $(S \subseteq N)$ is chosen. The dual game (N, c^*) of a cost game (N, c) is the *cost game* defined by $c^*(S) = c(N) - c(N \setminus S)$. So the dual game assigns to a coalition S *the marginal cost* of coalition S, if S joins the coalition $N \setminus S$. If $S \subseteq N$, the game (N, u_S^*) is the simple game with $u_S^*(T) = 1$ if and only if $S \cap T \neq \emptyset$. To define a weighted Shapley value we need a positive weight function on the universe U, i.e. $w: U \to IR_{++}$, and a collection $\mathcal{S} = (S_j)_{j \in I}$ of coalitions such that

1. $I \subseteq IN$, and
2. $S_j \subseteq U$, $\cup_{j \in I} S_j = U$, $S_j \cap S_k = \emptyset$ for $j, k \in I$.

For any coalition $\emptyset \neq T \subset U$ we define $T^0 := T \cap S_i$ where $T \cap S_i \neq \emptyset$ and $T \cap S_j = \emptyset$ for $j < i$.

The *$(w, \mathcal{S})$-weighted Shapley value* $\phi^{w,\mathcal{S}}$ is defined to be the linear solution concept given by

$$\phi_i^{w,\mathcal{S}}(N, u_S^*) = \begin{cases} w(i)/w(S^0), & \text{if } i \in S^0 \\ 0, & \text{otherwise} \end{cases}$$

The weighted Shapley value incorporates the idea of a 'hierarchical system' (like the caste system in traditional Hinduism) as well as the idea of 'measures of importance'. The coalition T^0 are the representatives of the 'highest caste' among the members of T.

The (symmetric) Shapley value ϕ (see Shapley, 1953) is obtained as a weighted Shapley value if all weights coincide and the index set I is a singleton. The first theorem of this section states the first-player consistency of the weighted Shapley value.

Theorem 3.1. *The* $(w, \mathcal{S})$*-weighted Shapley value* $\phi^{w,\mathcal{S}}$ *is first-player consistent on the set of airport games and its generating function* $g = g_{w,\mathcal{S}}$ *is defined by*

$$g(N, C) = \begin{cases} \dfrac{w([1])}{w(N^0)} C([1]), & \text{if } [1] \in N^0 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Let (N, c) be the game with $(N, C) \in \mathcal{A}$. Then

$$(N, c) = C[1](N, u^*_N) + \sum_{1 \le i < n} (C[i+1] - C[i])\ (N, u^*_{([i+1], \ldots, [n])}).$$

Therefore

$$x = \phi^{w,\mathcal{S}}_{[1]}(N, c) = C[1]\, \phi^{w,\mathcal{S}}_{[1]}(N, u^*_N) = \begin{cases} C[1] \dfrac{w[1]}{w(N^0)}, & \text{if } [1] \in N^0 \\ 0, & \text{otherwise} \end{cases}$$

holds true by the definition of $\phi^{w,\mathcal{S}}$. Hence, if the weighted Shapley value has a generating function, it is the function mentioned in the theorem. The game $(\bar{N}, \bar{c})$ associated with the reduced problem $(\bar{N}, \bar{C})$, i.e. $\bar{N} = N \backslash [1]$ and $\bar{C} = C^{[1],x}$, is

$$(\bar{N}, \bar{c}) = (C[2] - x)(\bar{N}, u^*_{([2], \ldots, [n])}) + \sum_{2 \le i < n} (C[i+1] - C[i])\ (\bar{N}, u^*_{([i+1], \ldots, [n])}).$$

It is clear from the definition of the weighted Shapley value that

$$\phi^{w,\mathcal{S}}(\bar{N}, u^*_S) = \phi^{w,\mathcal{S}}(N, u^*_S)_{\bar{N}} \text{ if } S \subseteqq \bar{N}.$$

So we are left to prove that

$$(C[2] - x)\, \phi^{w,\mathcal{S}}(\bar{N}, u^*_{\bar{N}}) = C[1]\, \phi^{w,\mathcal{S}}(N, u^*_N)_{\bar{N}} + (C[2] - C[1])\ \phi^{w,\mathcal{S}}(N, u^*_{([2], \ldots, [n])})_{\bar{N}},$$

or equivalently

$$(C[1] - x)\, \phi^{w,\mathcal{S}}(\bar{N}, u^*_{\bar{N}}) = C[1]\, \phi^{w,\mathcal{S}}(N, u^*_N)_{\bar{N}}.$$

There are three cases to be considered:

Case (a). $[1] \not\in N^0$.

Then $x = 0$ and $N^0 = \bar{N}^0$. The equality follows from the definition of $\phi^{w,\mathcal{S}}$.

Case (b). N^0 consists of [1] only.

Then $x = C[1]$ and we have to prove that $\phi^{w,\mathscr{S}}(N, u_N^*)_{\bar{N}} = 0$. This is clear from the definition of $\phi^{w,\mathscr{S}}$.

Case (c). $[1] \in N^0$ and N^0 contains also other players.

Then $\bar{N}^0 = N^0 \backslash [1]$. For $i \in \bar{N}^0$ we have to prove that

$$(C[1] - x)\ \frac{w\,[i]}{w\,(\bar{N}^0)} = C[1]\frac{w\,[i]}{w\,(N^0)}.$$

This can be done by substituting the expression for x. □

3.2. The nucleolus

The next result is about the first-player consistency of the nucleolus of airport games. For a cost game (N, c), $x \in X(N, c) = \{x \in IR^N | x(N) = c(N)\}$ and a coalition $S \subseteqq N$ let

$$e(S, x, c) = x(S) - c(S)$$

denote the *excess* of S at x (with respect to c). The *prenucleolus* of (N, c) is the unique member $\nu(N, c)$ of the set

$$\{x \in X(N, c) | \theta(e(S, x, c))_{S \subseteqq N} \leq_{lex} \theta(e(S, y, c))_{S \subseteqq N} \text{ for } y \in X(N, c)\}.$$

Here $\theta: IR^{2^N} \to IR^{2^n}$ orders the coordinates of a vector in a weakly *decreasing* way. Note that in the case of an airport game the prenucleolus is automatically individually rational (i.e. $\nu_i(N, c) \leq c(i)$ for $i \in N$) by concavity of (N, c), thus it is the *nucleolus* (see Schmeidler, 1969).

It is well-known (see Sobolev, 1975) that the prenucleolus satisfies the *reduced game property*, i.e., if $x = \nu(N, c)$ and $(S, \bar{c})$ is the *reduced game of (N, c)* in the sense of Davis and Maschler (1965) defined by

$$\bar{c}\,(T) = \begin{cases} c(N) - x(N \backslash S), & \text{if } T = S \\ 0, & \text{if } T = \emptyset \\ \min_{Q \subset N \backslash S} \left[c(T \cup Q) - x(Q) \right], & \text{otherwise} \end{cases},$$

then $\nu\,(S, \bar{c}) = \nu\,(N, c)_S$ for all coalitions $\emptyset \neq S \subseteqq N$.

Note that the associated game of a reduced airport problem is the (Davis–Maschler)-reduced game of the game associated with the original airport problem. In Littlechild and Thompson (1977) the nucleolus of airport games has been determined. From the results of this paper one can deduce that

$$\nu_{[1]}(N, C) = \min_{i<n} \frac{C[i]}{i+1}$$

in case $|N| > 1$. These observations imply directly the following result.

Theorem 3.2. *The nucleolus is first-player consistent on the class of airport problems and the generating function* $g = g^{\nu}$ *is defined by*

$$g(N, C) = \begin{cases} \min_{i<n} \dfrac{C[i]}{i+1}, & \text{if } n \geq 2 \\ C[1], & \text{if } n = 1 \end{cases}$$

for $(N, C) \in \mathcal{A}$.

3.3. The modified nucleolus

For the modified nucleolus we proceed differently. We give a formula for a rule and prove that it is a first-player consistent rule. The next step will be that we prove that this rule coincides with the modified nucleolus and the prenucleolus of the dual game.

If (N, C) is an airport problem, we can do the following: every player pays the total cost of the plan he is supporting and later on he obtains a discount λ in order to make the solution efficient. If the discount makes a player's payment negative, he pays nothing. So more formally, $\sigma^m(N, C)_i = (C(i) - \lambda)_+ = \max(0, C(i) - \lambda)$ and λ is chosen in such a way that σ^m is efficient . If C is not identically zero, then the number λ is uniquely determined by this condition. If $C = 0$, then we define $\lambda = 0$.

Remark 3.3. *Let $(N, C) \in \mathcal{A}$ satisfy $C \neq 0$ and $n \geq 2$. Then the finite sequence $(\lambda_i)_{i=1}^{n-1}$, defined by $\lambda_i = \Sigma_{j=i}^{n-1} C[j]/(n+1-i)$ is unimodal, i.e. there is $k \in \{1, \ldots, n-1\}$ such that $\lambda_i < \lambda_{i+1}$ for $i = 1, \ldots, k-1$ and $\lambda_j \geq \lambda_{j+1}$ for $j = k, \ldots, n-2$. Moreover, with $\lambda = \lambda_k$ the rule σ^m is determined by $\sigma_j^m(N, C) = (C(j) - \lambda)_+$ for $j \in N$. The straightforward proof is left to the reader.*

Theorem 3.4. *The rule σ^m on airport problems is first-player consistent.*

Proof. Let (N, C) be an airport problem with $n \geq 2$ and $C \neq 0$. Let $x = \sigma^m(N, C)_{[1]} = (C[1] - \lambda)_+$ be the payment by the first player [1] and consider the reduced problem $(\bar{N}, \bar{C})$. Let $\bar{\lambda}$ be the solution of the equation $\Sigma_{j \geq 2}(C[j] - x - t)_+ = C[n] - x$.

If we substitute $t = \lambda - x$ in the function $\bar{q}: t \to \Sigma_{j \geq 2}(C[j] - t)_+$, we obtain $\Sigma_{j \geq 2}(C[j] - \lambda)_+ = C[n] - (C[1] - \lambda)_+ = C[n] - x$. The first equality follows from the definition of λ. So $t = \lambda - x$ is the unique solution of $\bar{q}(t) = \bar{C}[n]$, i.e. $\bar{\lambda}$. Then the first-player consistency of σ^m follows. □

The next step is to prove that the first-player consistent rule σ^m assigns to each airport problem the modified nucleolus of the associated airport game. First we repeat the definition of the modified nucleolus (Sudhölter 1996, 1997).

When the nucleolus is an attempt to make the excesses of cost games lexicographically as small as possible, the modified nucleolus tries to make the differences between the excesses lexicographically as small as possible.

To define the modified nucleolus we need the *bi-excess* of a pair of coalitions, defined by

$$be(S, T, x, c) = e(S, x, c) - e(T, x, c)$$

for a game (N, c), coalitions $S, T \subseteq N$, and $x \in X(N, c)$. The second ingredient to define the modified nucleolus is the coordinate ordering map Θ that orders the coordinates of a vector from $\mathbf{R}^{2^N} \times \mathbf{R}^{2^N}$ in a weakly *decreasing* way. The *modified nucleolus* $\psi(N, c)$ *of a game* (N, c) is the unique member of the set

$$\{x \in X(N, c) | \Theta((be(S, T, x, c))_{S,T \subseteqq N} \leq_{lex} \Theta((be(S, T, y, c))_{S,T \subseteqq N} \text{ for } y \in X(N, c)\}.$$

The modified nucleolus is an example of a general nucleolus studied in Maschler et al. (1992). It shares several properties with the (pre)nucleolus (cf. Sudhölter, 1997). But it has also two properties in common with the Shapley value. The first property is *self-duality*, i.e. the modified nucleolus of the dual cost game is equal to the modified nucleolus of the original game. Another related property that the modified nucleolus shares with the Shapley value is the independence of the *interpretation of the game as profit game or cost game*.

The next theorem states that the modified nucleolus of an airport game is the solution σ^m we introduced before. This means that the modified nucleolus is first-player consistent. As a side result we will see that the modified nucleolus coincides with the prenucleolus of the dual cost game (also called the antinucleolus of the original game).

Theorem 3.5. $\psi(N, C) = \sigma^m(N, C)$ *for every airport problem* (N, C).

Proof. Let (N, C) be an airport problem, $x = \sigma^m(N, C)$, and (N, c) the associated airport game. In order to prove that $\psi(N, c) = x$ it can be assumed without loss of generality that $n \geq 2$ and $C \neq 0$. Let $K = \{j \in N | x_j = 0\}$ and $G = N \backslash K$, thus $K = \{[1], \ldots, [k]\}$ for some $k = 1, \ldots, n-1$. Let λ be the number defined in Remark 3.3. If $|G| \leq 1$, then $C[1] = \cdots = C[n-1] = 0$. The associated airport game assigns the value $C[n]$ to every coalition containing $[n]$. Then it is clear that the modified nucleolus equals $\psi(N, C) = C[n]e_{[n]}$ because all excesses and therefore all bi-excesses vanish with respect to this point. The other equality $C[n]e_{[n]} = \sigma_m(N, C)$ follows trivially from the definition of σ_m.

Therefore we assume $|G| \geq 2$. For the excesses $e(S, x, c)$ the following properties are obvious:

$$e(S, x, c) = 0, \text{ if } G \subseteqq S \text{ or } S = \emptyset \tag{3.1}$$

$$e(S, x, c) = -\lambda, \text{ if } |S \cap G| = 1 \tag{3.2}$$

$$-\lambda \leq e(S, x, c) \leq 0 \text{ for } S \subseteqq N \tag{3.3}$$

Therefore $\max_{S,T \subseteqq N} be(S, T, x, c) = \lambda$ and it suffices to show for every $y \in X(N, c)$ satisfying $\min_{S \subseteqq N} e(S, y, c) \geq -\lambda$ that $y = x$ holds true. Let

$$\mathcal{M} = \{(j) | j \in G\} \cup \{(i, j) | i \in K, j \in G\} \cup \{K \cup j \,|\, j \in G\}.$$

Then $e(S, x, c) = -\lambda$ for all $S \in \mathcal{M}$ and $\mathcal{M}$ is *balanced*, i.e. there are positive coefficients $\delta_S > 0$ such that

$$\sum_{S \in \mathcal{M}} \delta_S 1_S = 1_N,$$

where 1_S denotes the indicator function of S considered as element of the Euclidean space IR^N. Taking the inner product with y and x gives

$$y(N) = \sum_{S\in\mathcal{M}} \delta_S y(S) \geq \sum_{S\in\mathcal{M}} \delta_S(c(S) - \lambda) = \sum_{S\in\mathcal{M}} \delta_S x(S) = x(N).$$

We conclude $e(S, y, c) = -\lambda$ for $S \in \mathcal{M}$ by efficiency of y. The proof is finished by the observation that $\{1_S | S \in \mathcal{M}\}$ contains a basis of IR^N. □

Note that, in fact, in the proof of Theorem 3.5 the smallest excess is maximized. Therefore, we also have the following result.

Corollary 3.6. *The modified nucleolus of an airport game coincides with the nucleolus of the dual of the airport game.*

3.4. The τ-value

Example 3.7. (*The τ-value is not first-player consistent for airport games.*)

First we recall the definition of the τ-value (Tijs, 1981). If (N, c) is a cost game, we define the *marginal vector* $M(N, c)$ by $M(N, c)_i := c(N) - c(N \backslash i)$, $(i \in N)$. If the game (N, c) has a nonempty core, $\Sigma_{i\in N} M(N, c)_i \leq c(N)$. The remaining costs are measured by the vector $m(N, c)$ with coordinates

$$m(N, c)_i := \min_{S: i\in S} \left[c(S) - \sum_{j\in S, j\neq i} M(N, c)_j \right].$$

For cost games with a nonempty core $\Sigma_{i\in N} m(N, c)_i \geq c(N)$. The *τ-value of a cost game* is the unique efficient point on the line segment between $M(N, c)$ and $m(N, c)$. For *concave cost games* we have $m(N, c)_i = c(i)$ for all players $i \in N$. The next example shows that the τ-value is not first-player consistent.

Let $N = \{1, \ldots, 4\}$ and $C(1) = C(2) = 1$ and $C(3) = C(4) = 2$. The τ-value can be computed as $\tau(N, C) = (1, 1, 2, 2)/3$, whereas $\tau(N\backslash 1, C^{1,1/3}) = (10, 25, 25)/36$.

4. Monotonicity properties

A rule σ on the set $\mathcal{A}$ of airport problems satisfies

1. **fair ranking:** if, for $(N, C) \in \mathcal{A}$ and $i, j \in N$ satisfying $C(i) \leq C(j)$,

$$\sigma_i(N, C) \leq \sigma_j(N, C);$$

2. **monotonicity in costs:** if, for (N, C), $(N, C') \in \mathcal{A}$ and $i \in N$ satisfying $C'(i) \geq C(i)$ and $C'(j) = C(j)$ for $j \in N\backslash i$

$$\sigma_i(N, C') \geq \sigma_i(N, C);$$

3. **population monotonicity:** if, for $(N, C) \in \mathscr{A}$ and $\emptyset \neq S \subseteqq N$

$$\sigma(N, C)_S \leq \sigma(S, C_S);$$

4. **strong monotonicity:** if, for $(N, C),(N, C') \in \mathscr{A}$ and $i \in N$ satisfying $C'(i) - C'(j) \geq C(i) - C(j)$ for $j < i$ and $C'(i) \geq C(i)$

$$\sigma_i(N, C') \geq \sigma_i(N, C).$$

These properties have generalizations in the game theoretical context. Let $\tilde{\sigma}$ be a single valued solution concept on the set $\Gamma = \Gamma_U$ of cost games (N, c) with $N \subseteqq U$. Moreover, let σ denote its restriction to $\mathscr{A}$. If $\tilde{\sigma}$ preserves desirability in the sense of Maschler and Peleg (1966), then σ satisfies fair ranking. If $\tilde{\sigma}$ satisfies *coalitional monotonicity* (i.e., if $(N, c),(N, c') \in Ca$ and $i \in N$ satisfy $c(S) = c'(S)$ for all $S \subseteqq N \backslash i$ and $c'(S) \geq c(S)$ for all $S \subseteqq N$ with $i \in S$, then $\tilde{\sigma}_i(N, c') \geq \tilde{\sigma}_i(N, c)$), then σ satisfies monotonicity in costs. *Population monotonicity* of $\tilde{\sigma}$ is the straightforward generalization of population monotonicity of σ. If $\tilde{\sigma}$ satisfies *strong monotonicity* (i.e., if (N, c), $(N, c') \in Ca$ satisfy $c'(S \cup i) - c'(S) \geq c(S \cup i) - c(S)$ for all $S \subseteqq N$ and some $i \in N$, then $\tilde{\sigma}_i(N, c') \geq \tilde{\sigma}_i(N, c)$), then σ satisfies strong monotonicity.

Some well-known results are the following assertions.

1. The Shapley value, the nucleolus, and the modified nucleolus preserve desirability on Γ, thus satisfy fair ranking on $\mathscr{A}$ (see Maschler and Peleg, 1966 and Sudhölter, 1997).
2. Every weighted Shapley value satisfies strong monotonicity on Γ, thus on $\mathscr{A}$ (see Young, 1985).
3. The nucleolus on $\mathscr{A}$ is population monotonic (see Sönmez, 1993).

Lemma 4.1.

1. *The nucleolus on $\mathscr{A}$ satisfies monotonicity in costs.*
2. *The modified nucleolus on $\mathscr{A}$ satisfies monotonicity in costs.*

Proof. Let (N, C), $(N, C') \in \mathscr{A}, [i] \in N$ such that $C'[i] \geq C[i]$, $C'[j] = C[j]$ for $[j] \in N \backslash [i]$. If $n = 1$, both assertions follows from efficiency. Therefore we assume that $n \geq 2$ and that both assertions are valid for airport problems with strictly less players.

1. Let $x = \nu(N, C)$ and $x' = \nu(N, C')$. Then $g^{\nu}(N, C) = \frac{C[j]}{j+1}$ for some $j = 1, \ldots, n-1$ by Theorem 3.2. If $[j] < [i]$, then $g^{\nu}(N, C) = g^{\nu}(N, C')$ (by the same theorem), hence the proof follows from first-player consistency and the induction hypothesis. If $[j] \geq [i]$, one has $x'_{[i]} \geq x'_{[1]}$ by fair ranking, $x'_{[1]} \geq x_{[1]}$ by Theorem 3.2 and $x_{[1]} = \cdots = x_{[i]}$ follows from Littlechild and Thompson (1977).
2. Let $y = \psi(N, C)$ and $y' = \psi(N, C')$. By Theorem 3.5 $y_j = (C(j) - \lambda)_+$ and $y'_j = (C'(j) - \lambda')_+$ for $j \in N$ and some $\lambda, \lambda' \geq 0$ satisfying $\lambda \leq C[n], \lambda' \leq C'[n]$. Therefore, by efficiency,

$$\sum_{j\in N} (C\,(j)\text{-}\lambda)_+ = C[n]$$

and

$$\sum_{j\in N} (C'(j) - \lambda')_+ = C'[n].$$

For $i = n$, we find $\lambda' \le \lambda$. If $i < n$, $(C'[i] - \lambda')_+ < (C[i] - \lambda)_+$ would imply that

$$\sum_{j\in N} (C'(j) - \lambda')_+ < \sum_{j\in N} (C\,(j) - \lambda)_+,$$

a contradiction. □

Lemma 4.2. *The modified nucleolus on $\mathscr{A}$ satisfies population monotonicity.*

Proof. Let $(N, C) \in \mathscr{A}$ satisfy $n \ge 2$. Moreover, let $[i] \in N$, $x = \psi(N, C)$, and $\bar{x} = \psi(\bar{N}, C_{\bar{N}})$, where $\bar{N} = N\backslash[i]$. It suffices to show that $x_{\bar{N}} \le \bar{x}$. Choose $0 \le \lambda \le C[n] = C(l^N)$ and $0 \le \bar{\lambda} \le C(l^{\bar{N}})$ such that $x_j = (C(j) - \lambda)_+$ for $j \in N$ and $\bar{x}_j = (C(j) - \bar{\lambda})_+$ for $j \in \bar{N}$. We distinguish two cases.

1. $i \ne n$ or $i = n$ and $C[n-1] = C[n]$. Then $\bar{\lambda} \le \lambda$ by efficiency.
2. $i = n$ and $C[n] > C[n-1]$. As x belongs to the core by Theorem 2.4, we have $x(\bar{N}) \le C[n-1]$. From efficiency we have $\bar{x}(\bar{N}) = C[n-1]$. Therefore, $x(\bar{N}) \le \bar{x}(\bar{N})$ and $\bar{\lambda} \le \lambda$. □

Three-person airport problems show that the nucleolus and the modified nucleolus do not satisfy strong monotonicity on $\mathscr{A}$.

5. Axiomatizations

This section is devoted to axiomatizations of the nucleolus, the modified nucleolus, and the Shapley rule on $\mathscr{A}$. In the following definition we give two ways to extend the concepts of *reduced airport problem* and *consistency* in case a player $[i] \ne [1]$ leaves the game after paying an amount x. The player's payment is subtracted from the cost $C[i]$. In case of a ν-reduced airport problem the last part of $C[i]$ is diminished. For a ψ-reduced airport problem the first part of $C[i]$ is diminished.

Definition 5.1. *A rule σ on $\mathscr{A}$ satisfies*

1. **the equal treatment property**, if $\sigma_i(N, C) = \sigma_j(N, C)$ for $(N, C) \in \mathscr{A}$ and $i, j \in N$ satisfying $C(i) = C(j)$.
2. **covariance**, if
   2.1. $\sigma(N, \alpha C) = \alpha\sigma(N, C)$ for $(N, C) \in \mathscr{A}, \alpha > 0$, and
   2.2. whenever $(N, C), (N, C') \in \mathscr{A}$ satisfy $C[j] = C'[j]$ for $[j] \in N\backslash[n]$, and $C'[n] = C[n] + y$ with $y \ge 0$, then

$$\sigma(N, C') = \sigma(N, C) + ye_{[n]},$$

where $e_{[n]}$ denotes the $[n]$-th unit vector;

3. **ν-consistency**, if for $(N, C) \in \mathscr{A}$ with $n \geq 2$ and $[i] \in N$ the following conditions hold:
   3.1. $(\bar{N}, \bar{C}) = (N \backslash [i], {}^{\nu}C^{[i],x})$, defined by
   $$x = \sigma_{[i]}(N, C), \qquad \bar{C}[j] = \begin{cases} \min\{C[j], C[i] - x), & \text{if } j < i \\ C[j] - x, & \text{if } j > i\text{'} \end{cases}$$
   belongs to $\mathscr{A}$. In this case $(\bar{N}, \bar{C})$ is called the **ν-reduced airport problem** of (N, C) with respect to $[i]$ and x.
   3.2. $\sigma(\bar{N}, \bar{C}) = x_{\bar{N}}$.
4. **ψ-consistency**, if for $(N, C) \in \mathscr{A}$ with $n \geq 2$ and $[i] \in N \backslash [n]$ the following conditions hold:
   4.1. $(\bar{N}, \bar{C}) = (N \backslash [i], {}^{\psi}C^{[i],x})$, defined by
   $$x = \sigma_{[i]}(N, C), \qquad \bar{C}[j] = \begin{cases} (C[j] - x)_+, & \text{if } j < i \\ C[j] - x, & \text{if } j > i\text{'} \end{cases}$$
   belongs to $\mathscr{A}$. In this case $(\bar{N}, \bar{C})$ is called the **ψ-reduced airport problem** of (N, C) with respect to $[i]$ and x.
   4.2. $\sigma(\bar{N}, \bar{C}) = x_{\bar{N}}$.

Remark 5.2.

1. *Every ν-consistent and every ψ-consistent rule is first-player consistent, because the corresponding definitions coincide for $[i] = [1]$.*
2. *If a single valued solution concept $\tilde{\sigma}$ on Γ_U satisfies* **covariance** (*i.e.*, $\tilde{\sigma}(N, \alpha c + \beta) = \alpha\sigma(N, c) + \beta$ *for* $(N, c) \in \Gamma_U$, $\alpha > 0$, $\beta \in IR^N$), *then its restriction to $\mathscr{A}$ satisfies covariance.*
3. *If a single valued solution concept $\tilde{\sigma}$ on Γ_U satisfies* **reasonableness** (*i.e.*,
   $$\min\{c(S \cup i) - c(S) | S \subseteqq N \backslash i\} \leq \tilde{\sigma}_i(N, c) \leq \max\{c(S \cup i) - c(S) | S \subseteqq N \backslash i\}$$
   for $i \in N$ and $(N, c) \in \Gamma_U$) and the reduced game property, then its restriction to $\mathscr{A}$ satisfies automatically ν-consistency, since the associated game of a ν-reduced airport problem is the (Davis–Maschler)-reduced game of the game associated with the original airport problem. It is well-known that the prenucleolus has these properties.

This remark directly implies the following result.

Corollary 5.3. *The nucleolus rule on $\mathscr{A}$ satisfies ν-consistency.*

Theorem 5.4. *The modified nucleolus rule on $\mathscr{A}$ satisfies ψ-consistency.*

Proof. Let $(N, C) \in \mathscr{A}$ satisfy $n \geq 2$ and let x denote $\psi(N, C)$, i.e. $x_j = (C(j) - \lambda)_+$ for some $\lambda \geq 0$. Let $[i] \in N\backslash[n]$ and define $\bar{C} = {}^{\psi}C^{[i],x_{[i]}}$. Then $(N\backslash[i], \bar{C})$ is an airport problem by reasonableness and fair ranking. We have to find a number $\bar{\lambda}$ satisfying the equality

$$\sum_{[j] \in N\backslash[i]} (\bar{C}[j] - \bar{\lambda})_+ = \bar{C}[n] \tag{5.1}$$

and to prove that $(\bar{C}[j] - \bar{\lambda})_+ = (C[j] - \lambda)_+$ for $j \in N\backslash[i]$. We can rewrite equality 5.1 as:$\Sigma_{[j] \in N\backslash[i]}(C[j] - x_{[i]} - \bar{\lambda})_+ = C[n] - x_{[i]}$. If $\bar{\lambda}$ is defined by $\bar{\lambda} := \lambda - x_{[i]}$, then we have a solution of 5.1, because $x_{[j]} = (C[j] - \lambda)_+$ for all $j \neq i$. $\square$

Theorem 5.5.

1. *The nucleolus is the unique rule on $\mathscr{A}$ that satisfies the equal treatment property, covariance, and ν-consistency.*
2. *The modified nucleolus is the unique rule on $\mathscr{A}$ that satisfies the equal treatment property, covariance, and ψ-consistency.*

Proof.

1. It is well-known that the prenucleolus on Γ_U satisfies the 'equal treatment property' for games, which is a generalization of the property defined above, covariance, and consistency. Therefore, the nucleolus rule satisfies the desired properties. Let σ be a rule on $\mathscr{A}$ that satisfies the desired properties. Then σ satisfies efficiency by covariance and ν-consistency. The proof of Lemma 6.2.11 of Peleg (1988/89) can be copied literally. Note that **first-player consistency** and covariance guarantee this property. Moreover, it is a member of the prekernel (cf. Maschler et al. (1979). For a proof of this assertion in a more general framework see Peleg (1988/89). The prekernel is a singleton, hence coincides with the nucleolus, in this case because the corresponding airport games under consideration are concave (see Maschler et al. (1972).
2. The modified nucleolus satisfies the desired properties by Remark 3.3, Theorem 3.5, and Theorem 5.4. In order to show uniqueness let σ be a rule that satisfies the desired properties and, thus, efficiency (see 1). Then σ and ψ are the same for airport problems with less than three players by covariance and equal treatment.

   Assume that σ coincides with ψ for every airport problem with less than n players for some $n \geq 3$. Let $(N, C) \in \mathscr{A}$. By covariance we can assume without loss of generality that $C[n-1] = C[n]$. Let $x = \psi_{[n-1]}(N, C)$ and $y = \sigma_{[n-1]}(N, C)$. By ψ-consistency it suffices to show that $x = y$. Assume, on the contrary, that $x \neq y$. Let $(\bar{N}, \bar{C}_1)$ be the ψ-reduced airport problem with respect to $[n-1]$ and x and let $(\bar{N}, \bar{C}_2)$ be the ψ-reduced airport problem with respect to $[n-1]$ and y. By the induction hypothesis and ψ-consistency of σ and ψ we find:

$$\sigma(\bar{N}, \bar{C}_1) = \psi(\bar{N}, \bar{C}_1) = \psi_{\bar{N}}(N, C) \text{ and } \psi(\bar{N}, \bar{C}_2) = \sigma(\bar{N}, \bar{C}_2) = \sigma_{\bar{N}}(N, C).$$

   Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be the discount in $(\bar{N}, \bar{C}_1)$ and $(\bar{N}, \bar{C}_2)$, respectively. Then

$\psi_{[j]}(\bar{N}, \bar{C_1}) = (C[j] - x)_+ - \bar{\lambda_1})_+$ and $\psi_{[j]}(\bar{N}, \bar{C_2}) = (C[j] - y)_+ - \bar{\lambda_2})_+$ for $j \neq n-1$.

The equal treatment property and ψ-consistency of ψ give

$$x = \psi_{[n]}(N, C) = \psi_{[n]}(\bar{N}, \bar{C_1}) = (C[n] - x - \bar{\lambda_1}).$$

Equal treatment and ψ-consistency of σ give, together with the induction hypothesis:

$$y = \sigma_{[n]}(N, C) = \sigma_{[n]}(\bar{N}, \bar{C_2}) = \psi_{[n]}(\bar{N}, \bar{C_2}) = (C[n] - y - \bar{\lambda_2}).$$

Therefore, we find

$$\bar{\lambda_1} = C[n] - 2x \text{ and } \bar{\lambda_2} = C[n] - 2y.$$

If $x > y$, we find, for $j < n-1$

$$\psi_{[j]}(\bar{N}, \bar{C_1}) = ((C[j] - x)_+ + 2x - C[n])_+ \geq ((C[j] - y)_+ + 2y - C[n])_+ = \psi_{[j]}(\bar{N}, \bar{C_2}).$$

Then

$$\sum_{[j] \in \bar{N}} \psi_{[j]}(\bar{N}, \bar{C_1}) \geq \sum_{[j] \in \bar{N}} \psi_{[j]}(\bar{N}, \bar{C_2}).$$

By efficiency $C[n] - x \geq C[n] - y$ and, therefore, $x \leq y$.

If we start with $y > x$ we find by the same reasoning $y \leq x$. Only $x = y$ is possible. □

Remark 5.6. *The symmetric Shapley value is the unique rule on the set $\mathscr{A}$ of airport problems satisfying efficiency, the equal treatment property, and strong monotonicity.*

For a proof of this assertion, where $\mathscr{A}$ is replaced by the larger set Γ_U, refer to, e.g., Neyman (1989) or Peleg (1986). For the sake of completeness we present an analogous proof for the restricted set of airport problems.

Proof. For an airport problem (N, C) we introduce $h(N, C) := |\{[i] \mid C[i] < C[n]\}|$. The proof is by induction on $h(N, C)$. If $h(N, C) = 0$, we have a completely symmetric situation and $\sigma_{[i]} = \frac{C[n]}{n}$ for all players $[i] \in N$ by efficiency and the equal treatment property. The same is true for the Shapley value. If the theorem holds for $h(N, C) < k$ and (N, C) is an airport problem with $h(N, C) = k$, we have $C[1] \leq C[2] \leq \cdots \leq C[k] < C[k+1] = \cdots = C[n]$. If we define $\bar{C}$ by $\bar{C}[i] := \min\{C[i], C[k]\}$, we have by the induction hypothesis $\sigma(N, \bar{C}) = \phi(\bar{C})$. Notice that the marginals of the players $[i]$ with $i \leq k$ do not change if we go from C to $\bar{C}$ and therefore, $\sigma_{[i]}(N, C) = \sigma_{[i]}(N, \bar{C}) = \phi_{[i]}(N, \bar{C}) = \phi_{[i]}(N, C)$ for $i \leq k$ by strong monotonicity of σ and ϕ. The equal treatment property gives $\sigma_{[j]}(N, C) = \sigma_{[n]}(N, C)$ and $\phi_{[j]}(N, C) = \phi_{[n]}(N, C)$ for $j > k$ and efficiency gives $\sigma_{[n]}(N, C) = \phi_{[n]}(N, C)$. □

Some results are summarized in the following tables. The logical independence of the properties used in the characterizations of the nucleolus, modified nucleolus, and Shapley value is shown. For a weighted Shapley value $\phi^{w,\mathscr{S}}$ we assume that it is not symmetric, i.e. that it does not coincide with ϕ. The cardinality of the universe U is assumed to be at least 3. A '+' is the abbreviation for 'is satisfied', whereas '−' indicates that this property is not satisfied. We use the abbreviations 'EFF' for efficiency, 'ETP' for the equal treatment property, 'COV' for covariance, 'SM' for strong monotonicity, 'FR' for fair ranking, 'MC' for monotonicity in costs, 'PM' for population monotonicity, 'REAS' for reasonableness, 'FPC' for first-player consistency, 'ν-CONS' for ν-consistency, and 'ψ-CONS' for ψ-consistency. Many of the properties summarized in Table 1 are explicitly checked in the present paper. The remaining properties can easily be shown.

Theorem 5.5 and Remark 5.6 present axiomatizations of both nucleolus rules and the Shapley rule each by three properties, respectively. To show this we need rules that satisfy all characterizing properties except one. The rules $\sigma^1, \ldots, \sigma^6$ are defined as follows. Let $(N, C) \in \mathscr{A}$.

1.

$$\sigma^1_{[i]}(N, C) = \begin{cases} 0, & \text{if } C[i] < C[n] \\ \dfrac{C[n]}{p}, & \text{if } C[i] = C[n] \end{cases}, \text{ where } p = |\{i \mid C[i] = C[n]\}|.$$

2.

$$\sigma^2_{[i]}(N, C) = \begin{cases} 0, & \text{if } i \neq n \\ C[1], & \text{if } i = n = 1. \\ C[n] - C[n-1], & \text{otherwise} \end{cases}$$

3. σ^3 is the first-player consistent rule defined by the generating function $g^3 : \mathscr{A} \to IR^N$ that satisfies $g^3(N, C) = \min_{[k] \in N} (C[k]/k)$.
4.

$$\sigma^4_{[i]}(N, C) = \begin{cases} 0, & \text{if } i \neq n \\ C[n], & \text{if } i = n \end{cases}.$$

5. σ^5 is defined with the help of a distinguished potential player $\star \in U$. If $\star \notin N$, then $\sigma^5(N, C) = \nu(N, C)$. If $\star \in N$, then define $\sigma^5_i(N, C) = \nu_i(N \backslash \star, C_{N \backslash \star})$ for $i \neq \star$ and

Table 1
Properties

	EFF	ETP	COV	SM	FR	MC	PM	REAS	FPC	ν-CONS	ψ-CONS
ϕ	+	+	+	+	+	+	+	+	+	−	−
ν	+	+	+	−	+	+	+	+	+	+	−
ψ	+	+	+	−	+	+	+	+	+	−	+
τ	+	+	+	−	+	+	−	+	−	−	−
$\phi^{w,\mathscr{S}}$	+	−	+	+	−	+	+	+	+	−	−
ρ	+	+	+	−	+	+	−	−	+	−	−

Table 2
Independence

	EFF	ETP	COV	SM	ν-CONS	ψ-CONS	FPC
ϕ	+	+	+	+	−	−	+
ν	+	+	+	−	+	−	+
ψ	+	+	+	−	−	+	+
σ^1	+	+	−	−	−	+	+
σ^2	−	+	+	+	−	−	−
σ^3	+	+	−	−	+	−	+
σ^4	+	−	−	+	+	+	+
σ^5	+	−	+	−	+	−	+
σ^6	+	−	+	−	−	+	+

$\sigma^5_\star(N, C)$ to be the unique real number such that σ^5 satisfies efficiency. It should be remarked that the ν- or ψ-reduced airport problem with respect to the exceptional player coincides with the restriction of the airport problem to all other players.

6. σ^6 is defined analogously to σ^5 by interchanging the roles of ν and ψ.

It is straightforward to verify the properties of the rules on $\mathscr{A}$ summarized in the Table 2. Therefore ν, σ^2, σ^4 show the logical independence of efficiency, the equal treatment property, and strong monotonicity, whereas ϕ, σ^3, σ^5 or ϕ, σ^1, σ^6 show that the equal treatment property, covariance, and ν-consistency or ψ-consistency, respectively, are logically independent.

References

Davis, M., Maschler, M., 1965. The kernel of a cooperative game. Naval Research Logistics Quarterly 12, 223–259.

Dubey, P., 1982. The Shapley value as aircraft landing fees – revisited. Management Science 28, 869–874.

Kalai, E., Samet, D., 1988. Weighted Shapley Values, Essays in Honor of LS Shapley, 83–99.

Littlechild, S.C., Owen, G., 1973. A simple expression for the Shapley value in a special case. Management Science 20, 370–372.

Littlechild, S.C., Thompson, G.F., 1977. Aircraft landing fees: a game theory approach. Bell Journal of Economics 8, 186–204.

Maschler, M., Peleg, B., 1966. A characterization, existence proof and dimension bounds for the kernel of a game. Pacific Journal of Mathematics 18, 289–328.

Maschler, M., Peleg, B., Shapley, L.S., 1972. The kernel and bargaining set for convex games. International Journal of Game Theory 1, 73–93.

Maschler, M., Peleg, B., Shapley, L.S., 1979. Geometric properties of the kernel,nucleolus and related solution concepts. Mathematics of Operations Research 4, 303–338.

Maschler, M., Potters, J.A.M., Tijs, S.H., 1992. The general nucleolus and the reduced game property. International Journal of Game Theory 21, 85–105.

Milnor, J.W., 1952. Reasonable Outcomes for *n*-Person Games. The Rand Corporation, 196.

Moulin, H., Shenker, S., 1992. Serial cost sharing, Econometrica 60, 1009–1037.

Neyman, A., 1989. Uniqueness of the Shapley value. Games and Economic Behavior 1, 116–118.

Peleg, B., 1988/89. Introduction to the theory of cooperative games, Research Memoranda No. 81–88, Center for Research in Math. Economics and Game Theory, The Hebrew University, Jerusalem, Israel.

Peleg, B., 1986. On the reduced game property and its converse. International Journal of Game Theory 15, 187–200.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics 17, 1163–1170.

Shapley, L.S., 1953. A value for *n*-person games. In: Kuhn, H., Tucker, A.W. (Eds.), Contributions to the Theory of Games II, Princeton University Press, pp. 307–317.

Sobolev, A.I., 1975. The characterization of optimality principles in cooperative games by functional equations. In: Vorobiev, N.N. (Ed.), Mathematical Methods in Social Sciences 6, Academy of Sciences of the Lithuanian SSR, Vilnius, pp. 95–151, in Russian.

Sönmez, T., 1993. Population Monotonicity of the Nucleolus on a Class of Public Good Problems. Mimeo University of Rochester.

Sprumont, Y., 1990. Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic Behavior 2, 378–394.

Sudhölter, P., 1997. The modified nucleolus: properties and axiomatizations. International Journal of Game Theory 26, 147–182.

Sudhölter, P., 1996. The modified nucleolus as canonical representation of weighted majority games. Mathematics of Operations Research 21, 734–756.

Thomson, W., 1993. Population monotonic allocation rules, Rochester Center for Economic Research, Working Paper No. 375.

Tijs, S.H., 1981. Bounds for the core and the τ-value. In: Moeschlin, O., Pallaschke, D. (Eds.), Game Theory and Mathematical Economics, North Holland Publishing Company, Amsterdam, pp. 123–132.

Young, P., 1985. Monotonic solutions of cooperative games. International Journal of Game Theory 14, 65–72.