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On core stability, vital coalitions, and extendability

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ABSTRACT

If a TU game is extendable, then its core is a stable set. However, there are many TU games with a stable core that are not extendable. A coalition is vital if there exists some core element x such that none of the proper subcoalitions is effective for x . It is exact if it is effective for some core element. If all coalitions that are vital and exact are extendable, then the game has a stable core. It is shown that the contrary is also valid for matching games, for simple flow games, and for minimum coloring games.

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1. Introduction

The core of a cooperative game is called stable if it is a stable set in the sense of von Neumann and Morgenstern (1953). In this paper we restrict our attention to TU games. Several sufficient conditions for core stability may be found in the literature. For details see, e.g., van Gellekom et al. (1999). A weak and simple sufficient condition, introduced by Kikuta and Shapley (1986), is called *extendability*. A TU game is extendable if each core element of any subgame may be extended to a core element of the entire game. In contrast to Azrieli and Lehrer (2007), who showed that core largeness in the sense of Sharkey (1982) is equivalent to a strong version of extendability, the main part of the present paper is devoted to relaxing extendability in such a way that the modified extendability properties (1) are still sufficient conditions and (2) become necessary conditions for core stability when restricting the attention to some nontrivial important classes of games. We show that the game has a stable core if certain coalitions are extendable, namely those that are *vital* in the sense of Gillies (1959) and *exact* in the sense of Shapley (1971) and Schmeidler (1972). For some classes of games, e.g., for the class of symmetric games (see Biswas et al., 1999), necessary and sufficient conditions for core stability have been found. We show that vital–exact extendability is also a necessary condition for core stability for three important classes of games: Matching games, simple flow games, and minimum coloring games. Moreover, our approach enables us to reprove in a simple way two characterization results of Solymosi and Raghavan (2001) and of Bietenhader and Okamoto (2006).

The paper is organized as follows. In Section 2 the basic notation and the relevant definitions are presented and some relevant well-known results are recalled. Section 3 is devoted to three new extendability concepts. Theorem 3.3 states that the new variants of extendability are still sufficient for core stability. By means of examples it is shown that the modified

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conditions are weaker than extendability but still not necessary for core stability. Some properties of the new conditions are also discussed. Section 3.1 is devoted to the proof of Theorem 3.3 and in Section 3.2 it turns out that, if the vital and exact coalitions exhibit any of two additional properties (see Theorem 3.9 and Corollary 3.13), then the relaxed extendability condition is necessary for core stability. Section 4 is devoted to three classes of games that have been investigated in the literature. It is shown that the relaxed extendability condition is a necessary condition for core stability in these three cases. The theory developed so far enables us to extend to the class of matching games the well-known characterization result of assignment games with a stable core. Moreover, in the case of minimum coloring games, the well-known characterization result is generalized.

2. Preliminaries

In this section we recall definitions of some relevant concepts and well-known results that may be found in von Neumann and Morgenstern (1953) or Gillies (1959) unless otherwise specified.

A (cooperative TU) game is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \rightarrow \mathbb{R}$, $v(\emptyset) = 0$. Let (N, v) be a game. For $S \subseteq N$ we denote by \mathbb{R}^S the set of all real functions on S . So \mathbb{R}^S is the $|S|$ -dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D .) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x_i > y_i$ for all $i \in S$.

Let $X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ denote the set of Pareto optimal allocations (preimputations) of (N, v) . We use $x(S) = \sum_{i \in S} x_i$ ($x(\emptyset) = 0$) for every $S \in 2^N$ and every $x \in \mathbb{R}^N$. Additionally, x_S denotes the restriction of x to S , i.e. $x_S = (x_i)_{i \in S}$.

The core of (N, v) , $\mathcal{C}(N, v)$, is given by

$$\mathcal{C}(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \ \forall S \subseteq N\}.$$

The set of imputations of (N, v) , $I(N, v)$, is $I(N, v) = \{x \in X(N, v) \mid x_i \geq v(\{i\}) \ \forall i \in N\}$.

A coalition (in N) is a nonempty subset of N . A subgame of (N, v) is a game (T, v^T) where T is a coalition and $v^T(S) = v(S)$ for all $S \subseteq T$. The subgame (T, v^T) will also be denoted by (T, v) .

Let $x, y \in \mathbb{R}^N$ and $S \in 2^N \setminus \{\emptyset\}$. We say that x dominates y via S (at (N, v)), written $x \text{ dom}_S y$, if $x(S) \leq v(S)$ and $x_S \gg y_S$. Also, we define $x \text{ dom } y$, that is, x dominates y (at (N, v)), if there exists a coalition S in N such that $x \text{ dom}_S y$. Let $X \subseteq \mathbb{R}^N$. We say that X is internally stable (with respect to (w.r.t.) (N, v)) if for any $x \in X$ and $y \in \mathbb{R}^N$, $x \text{ dom } y$ implies that $y \notin X$. Moreover, X is externally stable (w.r.t. (N, v)) if for any $y \in I(N, v) \setminus X$ there exists $x \in X$ such that $x \text{ dom } y$. The set X is stable if it is internally and externally stable.

Note that $\mathcal{C}(N, v)$ is internally stable and that any externally stable set contains $\mathcal{C}(N, v)$. We say that (N, v) has a stable core if $\mathcal{C}(N, v)$ is stable, that is, externally stable, w.r.t. (N, v) . We also remark that, if $I(N, v) = \emptyset$, then $\emptyset = \mathcal{C}(N, v)$ is stable. Hence, we shall not further consider the case that $\sum_{i \in N} v(\{i\}) > v(N)$.

We now recall some relevant results. The proof of the well-known Proposition 2.1 is presented, because its statement will be used several times.

Proposition 2.1. (See Gillies, 1959.) Let (N, v) be a game such that $I(N, v) \neq \emptyset$. If (N, v) has a stable core, then, for each $i \in N$, there exists $x \in \mathcal{C}(N, v)$ such that $x_i = v(\{i\})$.

Proof. As (N, v) has a stable core and $I(N, v) \neq \emptyset$, $\mathcal{C}(N, v) \neq \emptyset$. Assume, on the contrary, that there exists $k \in N$ such that $x_k > v(\{k\})$ for all $x \in \mathcal{C}(N, v)$. As $\mathcal{C}(N, v)$ is a compact set, $t = \min\{x_k \mid x \in \mathcal{C}(N, v)\}$ exists so that $t > v(\{k\})$. Choose $x \in \mathcal{C}(N, v)$ with $x_k = t$, let $\varepsilon > 0$ satisfy $t - (|N| - 1)\varepsilon \geq v(\{k\})$, and define $y \in \mathbb{R}^N$ by $y_i = x_i + \varepsilon$ for all $i \in N \setminus \{k\}$ and $y_k = x_k - (|N| - 1)\varepsilon$. Then $y \in I(N, v) \setminus \mathcal{C}(N, v)$. Hence, there exist $z \in \mathcal{C}(N, v)$ and $\emptyset \neq T \subseteq N$ with $z \text{ dom}_T y$. For any $S \subseteq N \setminus \{k\}$, $y(S) = \varepsilon|S| + x(S) \geq x(S) \geq v(S)$. Hence, $k \in T$ so that $z_k \geq t = x_k$. As $y_{N \setminus \{k\}} \gg x_{N \setminus \{k\}}$, we conclude that $z(T) > x(T) \geq v(T)$ and the desired contradiction has been obtained. \square

The foregoing proposition has the following interesting consequence.³

Corollary 2.2. If the game (N, v) has a stable core, then any preimputation of (N, v) that is not in $\mathcal{C}(N, v)$ is dominated by some element of $\mathcal{C}(N, v)$, provided that $I(N, v) \neq \emptyset$.

In order to recall the Bondareva–Shapley theorem (see Bondareva, 1963 and Shapley, 1967) which gives necessary and sufficient conditions for the nonemptiness of the core, the following notation is useful. For $T \subseteq N$, denote by $\chi^T \in \mathbb{R}^N$ the characteristic vector of T , defined by

$$\chi_i^T = \begin{cases} 1, & \text{if } i \in T, \\ 0, & \text{if } i \in N \setminus T. \end{cases}$$

³ Corollary 2.2 may not hold for an arbitrary stable set. Indeed, if (N, v) is the three-person majority game, defined by $N = \{1, 2, 3\}$, $v(N) = v(S) = 1$, if $|S| = 2$, and $v(T) = 0$, if $|T| \leq 1$, then $X = \{(c, \frac{3}{4} - c, \frac{1}{4}) \mid 0 \leq c \leq \frac{3}{4}\}$ is a well-known stable set, but the preimputation $(1, 1, -1)$ is not dominated by an element of X .

A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* (over N) if positive numbers $\delta^S, S \in \mathcal{B}$, exist such that $\sum_{S \in \mathcal{B}} \delta^S \chi^S = \chi^N$. The collection $(\delta^S)_{S \in \mathcal{B}}$ is called a system of *balancing weights* for \mathcal{B} .

Theorem 2.3 (The Bondareva–Shapley Theorem). *Let (N, v) be a game. Then $\mathcal{C}(N, v) \neq \emptyset$ if and only if for each balanced collection \mathcal{B} over N and any system $(\delta^S)_{S \in \mathcal{B}}$ of balancing weights for \mathcal{B} , $\sum_{S \in \mathcal{B}} \delta^S v(S) \leq v(N)$.*

The foregoing theorem motivates calling a game (N, v) a *balanced game* if $\mathcal{C}(N, v) \neq \emptyset$. Note that (N, v) is *totally balanced* if, for any $\emptyset \neq S \subseteq N$, (S, v) is balanced. The *totally balanced cover* of (N, v) , (N, v^{tb}) , is given by

$$v^{tb}(S) = \max \left\{ \sum_{T \in \mathcal{B}} \delta^T v(T) \mid \begin{array}{l} \mathcal{B} \text{ is a balanced collection over } S \text{ and} \\ (\delta^T)_{T \in \mathcal{B}} \text{ is system of balancing weights for } \mathcal{B} \end{array} \right\} \forall S \subseteq N. \tag{2.1}$$

The formulation of a weak sufficient condition for core stability requires some notation. Let (N, v) be a game and let $\emptyset \neq S \subseteq N$. The coalition S is called *extendable* (w.r.t. (N, v)) if, for any $x \in \mathcal{C}(S, v)$, there exists $y \in \mathcal{C}(N, v)$ such that $x = y_S$. Moreover, (N, v) is *extendable* if all coalitions are extendable. The proof of the following well-known result is straightforward.

Theorem 2.4. (See Kikuta and Shapley, 1986.) *Any extendable game (N, v) has a nonempty stable core.*

3. Relaxing extendability

This section is organized as follows. The present part introduces conditions that are weaker than extendability. The main result of this section, Theorem 3.3, states that these new variants of extendability are sufficient conditions for core stability. Moreover, properties and relations of the new variants of extendability are presented. Section 3.1 is devoted to the proof of the main result and in Section 3.2 we show that certain assumptions on the structure of a game guarantee that the new conditions are necessary for core stability.

We now recall two possible properties of a coalition w.r.t. a game. Let (N, v) be a game and $\emptyset \neq S \subseteq N$. The coalition S is called *exact* (w.r.t. (N, v)) if there exists $x \in \mathcal{C}(N, v)$ such that $x(S) = v(S)$. In this case S is *effective* for x . If all coalitions are exact, then (N, v) is called *exact* (see Shapley, 1971 or Schmeidler, 1972). We say that a balanced game (N, v) is *exact extendable* if all exact coalitions are extendable. Moreover, S is called *vital* (w.r.t. (N, v)) if there exists $x \in \mathcal{C}(S, v)$ such that $x(T) > v(T)$ for⁴ all $T \in 2^S \setminus \{\emptyset, S\}$. We say that a balanced game (N, v) is *vital extendable* if all vital coalitions w.r.t. (N, v) are extendable.

Remark 3.1. There is a simple characterization of a vital coalition (see Gillies, 1959). Indeed, S is vital if and only if for any balanced collection \mathcal{B} over S , $S \notin \mathcal{B}$, and any system $(\delta^T)_{T \in \mathcal{B}}$ of balancing weights for \mathcal{B} , $\sum_{T \in \mathcal{B}} \delta^T v(T) < v(S)$.

Denote by $\mathcal{E}(N, v)$ the set of all coalitions S that are effective for x for all $x \in \mathcal{C}(N, v)$ or $S = \emptyset$, that is,

$$\mathcal{E}(N, v) = \{S \subseteq N \mid x(S) = v(S) \ \forall x \in \mathcal{C}(N, v)\}. \tag{3.1}$$

Let (N, v) be a balanced game. Then $\emptyset, N \in \mathcal{E}(N, v)$. Moreover, if $\mathcal{E}(N, v) = \{\emptyset, N\}$, then (N, v) may or may not have a stable core, provided that $|N| \geq 3$ (as simple examples show). Our analysis of, e.g., matching games (see Section 4.1), however, requires to consider also the case that $\mathcal{E}(N, v)$ may contain nonempty proper subsets of N .

Definition 3.2. Let (N, v) be a balanced game. A coalition $S \subseteq N$ is called *strongly vital–exact* (w.r.t. (N, v)) if S is vital and if there exists $x \in \mathcal{C}(N, v)$ such that

$$x(S) = v(S) \quad \text{and} \quad x(T) > v(T) \quad \text{for all } T \in 2^S \setminus (\{S\} \cup \mathcal{E}(N, v)). \tag{3.2}$$

The game (N, v) is *vital–exact extendable* if all strongly vital–exact coalitions are extendable.

Theorem 3.3. *Any balanced, vital–exact extendable game (N, v) has a stable core.*

Remark 3.4.

- (1) Note that vital–exact extendability is implied by the condition that every vital and exact coalition is extendable. The latter condition, however, is unnecessarily strong as Example 3.6 shows: Indeed, the game (N, v_2) is balanced and vital–exact extendable, but the coalition T is vital, exact, and not extendable.

⁴ Gillies (1959) introduced vital coalitions of at least two elements, whereas according to our definition singletons are always vital.

(2) On the other hand, note that a further relaxation of vital-exact extendability might be problematic; e.g., if Definition 3.2 of a strongly vital-exact coalition S is modified only inasmuch as “ $T \in 2^S \setminus (\{S\} \cup \mathcal{E}(N, v))$ ” is replaced by “ $T \in 2^S \setminus \{S, \emptyset\}$ ”, then the arising relaxation of vital-exact extendability may not be sufficient for core stability as shown by the following example: Let (N, v) be defined by $N = \{1, \dots, 4\}$, $v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = v(N) = 1$, $v(T) = 0$, otherwise. Then $\mathcal{C}(N, v) = \{(0, 0, \alpha, 1 - \alpha) \mid 0 \leq \alpha \leq 1\}$. Thus, all singletons are extendable. The remaining vital coalitions, i.e., $\{1, 3, 4\}$ and $\{2, 3, 4\}$, violate the aforementioned modified definition of strong vital-exactness, because $x_1 = v(\{1\}) = x_2 = v(\{2\}) = 0$ for $x \in \mathcal{C}(N, v)$. Clearly, (N, v) does not have a stable core.

Thus, Theorem 3.3 shows relations that may be summarized in the following diagram:

$$\begin{array}{ccccc} & & \text{exact extendability} & & \\ & \nearrow & & \searrow & \\ \text{extendability} & & & & \text{vital-exact extendability} \Rightarrow \text{core stability.} \\ & \searrow & & \nearrow & \\ & & \text{vital extendability} & & \end{array} \quad (3.3)$$

By means of examples we will show that none of the opposite implications of (3.3) is valid and that exact extendability may not imply vital extendability and vice versa. Moreover, there are balanced games that are vital-exact extendable and have nonextendable coalitions that are vital and exact.

Example 3.5. Let $N = \{1, \dots, 7\}$ and let (N, v_1) be defined as follows. Let $T = \{1, 2\}$, $T^i = \{2, i\}$ for $i = 3, 4, 5$, and $T^j = \{1, j\}$ for $j = 6, 7$, and let $v_1(N) = 16$, $v_1(T^k) = 4$ for all $k = 3, \dots, 7$, $v_1(T) = 1$, and for all other $S \subseteq N$, let $v_1(S) = 0$. Then $(3, 3, 2, 2, 2, 2, 2) \in \mathcal{C}(N, v_1)$ so that $\mathcal{E}(N, v_1) = \{\emptyset, N\}$. With

$$y^1 = (12, 4, 0, 0, 0, 0, 0), \quad y^2 = (0, 2, 2, 2, 2, 4, 4), \quad y^3 = (4, 0, 4, 4, 4, 0, 0)$$

note that $y^i \in \mathcal{C}(N, v_1)$ for $i = 1, 2, 3$. The coalition T is vital, but not exact. Indeed, let $y \in \mathcal{C}(N, v_1)$. As $y(T^k) \geq 4$, $k = 3, \dots, 7$,

$$y_i \geq 4 - y_2 \quad \forall i \in \{3, 4, 5\} \quad \text{and} \quad y_j \geq 4 - y_1 \quad \forall j \in \{6, 7\} \quad (3.4)$$

so that $16 = y(N) \geq 20 - y(T) - y_2$, that is, $y(T) \geq 2$. We conclude that a coalition $S \subsetneq N$ satisfying $v_1(S) > 0$ is exact if and only if it is one of the coalitions T^j , $j = 3, \dots, 7$, and that these coalitions are extendable. An exact coalition S with $v_1(S) = 0$ is also extendable, because $\mathcal{C}(S, v_1)$ is a singleton. Hence, (N, v_1) is exact extendable, but not vital extendable. Let (N, v'_1) be the game that differs from (N, v_1) only inasmuch as $v'_1(T) = 0$. Then (N, v'_1) is vital extendable (because T is not vital w.r.t. (N, v'_1)) and exact extendable, but T is still not extendable.

Example 3.6. Now, let (N, v_2) be the game that differs from (N, v_1) defined in Example 3.5 only inasmuch as $v_2(N) = 18$. Any singleton and any of the coalitions T^j , $j = 3, \dots, 7$, are still extendable which follows from the fact that $y^k + 2\chi^{(i)} \in \mathcal{C}(N, v_2)$ for any $k = 1, 2, 3$, and $i \in N$. Moreover, $z = (0, 1, 3, 3, 3, 4, 4)$ is the unique element in $\mathcal{C}(N, v_2)$ that satisfies $z(T) = v_2(T)$. Hence, T is vital and exact, but not strongly vital-exact. We conclude that (N, v_2) is vital-exact extendable, but neither exact extendable nor vital extendable. Now, if the worth of N is further increased, that is, let $0 < \varepsilon < 1$ and (N, v_3) differ from (N, v_2) only inasmuch as $v_3(N) = v_2(N) + \varepsilon$, then $(\varepsilon, 1 - \varepsilon, 3 + \varepsilon, 3 + \varepsilon, 3 + \varepsilon, 4 - \varepsilon, 4 - \varepsilon) \in \mathcal{C}(N, v_3)$ so that T is strongly vital-exact. Now, T is not extendable, because if $y \in \mathcal{C}(N, v_3)$ satisfies $y_2 = 0$, then $y_1 \geq 2 - \varepsilon > 1$, that is, $y(T) > v_3(T)$. Nevertheless, (N, v_3) has a stable core. Indeed, if $x \in I(N, v_3) \setminus \mathcal{C}(N, v_3)$, then two cases may occur. If $x(T^j) \geq 4$ for all $j = 3, \dots, 7$, then, by (3.4) applied to x , $x_2 + x(T) \geq 2 - \varepsilon$. As $x(T) < 1$, $x_2 > 1 - \varepsilon$ and $x_1 < \varepsilon$ so that x is dominated by some core element via T . In the other case there exists $\ell \in \{3, \dots, 7\}$ such that $x(T^\ell) < v(T^\ell)$ and extendability of T^ℓ guarantees that x is dominated by some core element.

Together with Example 4.4 (the game (N, v_4) discussed in Section 4) the foregoing examples show that the relations summarized in (3.3) are strict even if balancedness is assumed:

$$\begin{array}{ccccccc} & & & \text{vital extendability} & & & \\ & & & & \searrow & & \\ \text{core stability} & \xrightarrow{v_3} & \text{vital-exact extendability} & \xrightarrow{v_2} & v_4 & \xrightarrow{v_1} & v'_1 \text{ extendability.} \\ & & & \xrightarrow{v_2} & \text{exact extendability} & & \end{array}$$

The properties of the games (N, v_1) and (N, v_3) of Example 3.5 also show that neither “exact extendability” nor “vital-exact extendability” are strong prosperity properties in the sense of van Gellekom et al. (1999, Definition 2.1) who showed that “extendability” is a strong prosperity property. Note that in a similar way (indeed, a nonempty proper coalition in N is or is not vital regardless of the “prosperity” of N) it may be shown that “vital extendability” is a strong prosperity property.

An interesting invariance property shared by two of the new variants of “extendability” and by “core stability” is contained in the following statements. Let (N, v) be a balanced game and (N, v^{tb}) its totally balanced cover (see (2.1)):

- (1) (N, v) has a stable core $\iff (N, v^{tb})$ has a stable core.
- (2) (N, v) is vital extendable $\iff (N, v^{tb})$ is vital extendable.

(3) (N, v) is vital–exact extendable $\iff (N, v^{tb})$ is vital–exact extendable.

For a proof of (1) see van Gellekom et al. (1999, p. 220) who also show by means of Example 2 that there exists an extendable game whose totally balanced cover is not extendable. By (2.1), $\mathcal{C}(N, v) = \mathcal{C}(N, v^{tb})$. We conclude that a coalition is vital w.r.t. (N, v) iff it is vital w.r.t. (N, v^{tb}) . Moreover, if S is vital, then $v(S) = v^{tb}(S)$. Therefore, S is vital and exact w.r.t. (N, v) iff S is vital and exact w.r.t. (N, v^{tb}) . Hence, if S is vital and exact, then $\{x \in \mathcal{C}(N, v) \mid x(S) = v(S)\} = \{x \in \mathcal{C}(N, v^{tb}) \mid x(S) = v^{tb}(S)\}$. Again by (2.1), we may deduce from the foregoing equation that a coalition is strongly vital–exact w.r.t. (N, v) iff it is strongly vital–exact w.r.t. (N, v^{tb}) . Thus, (2) and (3) are valid. The totally balanced cover of (N, v_1) , (N, v_1^{tb}) , is not exact extendable. Indeed, it is straightforward to verify that $v_1^{tb}(\{1, 2, 3, 6\}) = 8$ and that $(1, 0, 4, 3) \in \mathcal{C}(\{1, 2, 3, 6\}, v_1^{tb})$. However, this vector is not the restriction of any element of $\mathcal{C}(N, v_1^{tb})$.

3.1. The proof of Theorem 3.3

We now prove two useful lemmata. Let (N, v) be a balanced game.

Lemma 3.7. *For any $x \in X(N, v) \setminus \mathcal{C}(N, v)$ there exists a strongly vital–exact coalition P such that $x(P) < v(P)$.*

Proof. By the definition of $\mathcal{E}(N, v)$ and the convexity of the core, there exists $x^0 \in \mathcal{C}(N, v)$ such that $x^0(S) > v(S)$ for all $S \in 2^N \setminus \mathcal{E}(N, v)$. For $\lambda \in \mathbb{R}$ denote $x^\lambda = \lambda x + (1 - \lambda)x^0$. As $\mathcal{C}(N, v)$ is convex and closed, there exists $\hat{\lambda}, 0 \leq \hat{\lambda} < 1$, such that

$$\lambda \geq 0 \text{ and } x^\lambda \in \mathcal{C}(N, v) \iff 0 \leq \lambda \leq \hat{\lambda}.$$

Then there exists $P \subseteq N$ such that $x(P) < v(P)$ and $x^{\hat{\lambda}}(P) = v(P)$. Hence, P is exact. Now, let P be minimal (w.r.t. inclusion) such that $x(P) < v(P)$ and $x^{\hat{\lambda}}(P) = v(P)$. We claim that

$$x^{\hat{\lambda}} \in \mathcal{C}(N, v), \quad x^{\hat{\lambda}}(P) = v(P), \quad \forall Q \in 2^P \setminus (\{P\} \cup \mathcal{E}(N, v)): x^{\hat{\lambda}}(Q) > v(Q).$$

In order to verify our claim, note that, by minimality of P ,

$$Q \subsetneq P, \quad x(Q) < v(Q) \implies x^{\hat{\lambda}}(Q) > v(Q). \tag{3.5}$$

Moreover, we observe that

$$Q \subsetneq P, \quad x(Q) > v(Q), \quad 0 < \lambda \leq 1 \implies x^\lambda(Q) > v(Q); \tag{3.6}$$

$$Q \subsetneq P, \quad x(Q) = v(Q), \quad Q \in \mathcal{E}(N, v), \quad 0 \leq \lambda \leq 1 \implies x^\lambda(Q) = v(Q); \tag{3.7}$$

$$Q \subsetneq P, \quad x(Q) = v(Q), \quad Q \notin \mathcal{E}(N, v), \quad 0 \leq \lambda < 1 \implies x^\lambda(Q) > v(Q). \tag{3.8}$$

By (3.5), (3.6), and (3.8), $x^{\hat{\lambda}} \in \mathcal{C}(N, v)$, $x^{\hat{\lambda}}(P) = v(P)$, $x^{\hat{\lambda}}(Q) > v(Q)$ for all $Q \in 2^P \setminus \mathcal{E}(N, v)$, $Q \neq P$, i.e., our claim. Hence, P is an exact coalition that satisfies (3.2). In order to show that P is vital–exact it remains to show that P is vital, i.e., it suffices to construct $y \in \mathbb{R}^N$ such that

$$y(P) = v(P), \quad y(Q) > v(Q) \quad \forall Q \in 2^P \setminus \{P, \emptyset\}.$$

By (3.7) there exists $\varepsilon > 0$, $\varepsilon \leq 1 - \hat{\lambda}$, such that $x^{\hat{\lambda} + \varepsilon}(Q) \geq v(Q)$ for all $Q \subsetneq P$. Then $d = v(P) - x^{\hat{\lambda} + \varepsilon}(P) > 0$. Now, with $y = x^{\hat{\lambda} + \varepsilon} + \frac{d}{|P|} \chi^P$ we observe that y has the desired properties, i.e., $y(P) = v(P)$ and $y(Q) > v(Q)$ for all $Q \in 2^P \setminus \{\emptyset, P\}$. \square

Lemma 3.8. *If (N, v) is vital–exact extendable and $x \in X(N, v) \setminus \mathcal{C}(N, v)$, then there exists a strongly vital–exact coalition S such that $x(S) < v(S)$ and $x(T) \geq v(T)$ for all $T \subsetneq S$.*

Proof. By Lemma 3.7 there exists a strongly vital–exact coalition P such that $x(P) < v(P)$. Let P be a minimal coalition that satisfies the foregoing conditions. Assume, on the contrary, that there exists $Q \subsetneq P$ such that $x(Q) < v(Q)$. Define

$$y = x + \frac{v(P) - x(P)}{|P \setminus Q|} \chi^{P \setminus Q}$$

and observe that $x \leq y$, $x(Q) = y(Q)$, and $y(P) = v(P)$. Hence $y_P \in X(P, v) \setminus \mathcal{C}(P, v)$. By Lemma 3.7 applied to (P, v) and y_P , there exists a strongly vital–exact coalition T w.r.t. (P, v) such that $y(T) < v(T)$ and, hence, $x(T) < v(T)$. As P is extendable, T is strongly vital–exact w.r.t. (N, v) so that the desired contradiction has been obtained. \square

Proof of Theorem 3.3. Let $z \in X(N, v) \setminus \mathcal{C}(N, v)$. By Lemma 3.8 there exists a strongly vital–exact $\emptyset \neq S \subseteq N$ such that $z(S) < v(S)$ and $z(T) \geq v(T)$ for all $T \subsetneq S$. Let $y \in \mathbb{R}^S$ be given by $y_i = z_i + \frac{v(S) - z(S)}{|S|}$. Then $y(S) = v(S)$ and $y \gg z$, hence $y(T) > v(T)$ for all $\emptyset \neq T \subsetneq S$. We conclude that $y \in \mathcal{C}(S, v)$. As S is extendable, there exists $x \in \mathcal{C}(N, v)$ such that $x_S = y$. Thus $x \text{ dom}_S z$. \square

3.2. Two consequences of Theorem 3.3

This subsection serves to show that all strongly vital–exact coalitions are extendable, if the set of strongly vital–exact coalitions exhibits a certain structure. We say that (N, v) has *disjoint antichains of strongly vital–exact coalitions* if, for all strongly vital–exact coalitions S and T , $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$ (that is, the elements of any antichain of the partially ordered set of strongly vital–exact coalitions, ordered by inclusion, are pairwise disjoint).

Theorem 3.9. *If (N, v) is a balanced game that has disjoint antichains of strongly vital–exact coalitions, then (N, v) has a stable core.*

Proof. Let S be a strongly vital–exact coalition. By Theorem 3.3 it suffices to show that S is extendable. To this extent let $x \in C(S, v)$. As S is exact, there exists $y \in C(N, v)$, $y(S) = v(S)$. Let $z \in \mathbb{R}^N$ be given by $z_S = x$ and $z_{N \setminus S} = y_{N \setminus S}$. We conclude that $z(N) = v(N)$, $z(T) = y(T) \geq v(T)$ for all $T \subseteq N \setminus S$ and all $S \subseteq T \subseteq N$, and $z(P) = x(P) \geq v(P)$ for all $P \subseteq S$. Hence, $z(Q) \geq v(Q)$ for all strongly vital–exact coalitions Q . By Lemma 3.7, $z \in C(N, v)$ and the proof is complete. \square

Balanced games that have disjoint antichains of strongly vital–exact coalitions may be constructed as follows. Let N be a finite nonempty set, let $x \in \mathbb{R}^N$, and let (N, v) satisfy $v(S) = x(S)$ for all $S \subsetneq N$ and $v(N) \geq x(N)$. Then the strongly vital–exact coalitions are the singletons and N provided that $v(N) > x(N)$. Hence (N, v) has the desired property. Now let $(N^1, v^1), \dots, (N^k, v^k)$ be k balanced games that have disjoint antichains of strongly vital–exact coalitions such that the N^ℓ are pairwise disjoint. With $N = \bigcup_{\ell=1}^k N^\ell$ let (N, v) be a game that satisfies $v(S) = \sum_{\ell=1}^k v^\ell(S \cap N^\ell)$ for all $S \subsetneq N$ and $v(N) \geq \sum_{\ell=1}^k v^\ell(N^\ell)$. Then (N, v) has the desired property.

The following theorem reveals some structure of the set of strongly vital–exact coalitions and will be used to show that vital–exact extendability is a necessary condition for core stability for the second class of games.

Theorem 3.10. *If (N, v) is a balanced game, then there exist a balanced collection $\mathcal{P} \subseteq \mathcal{E}(N, v)$ of strongly vital–exact coalitions w.r.t. (N, v) and a system $(\delta^P)_{P \in \mathcal{P}}$ of balancing weights for \mathcal{P} such that*

$$\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N).$$

Proof. Let (N, v) be balanced. We claim that the following statement is true:

$$R \in \mathcal{E}(N, v), \quad \emptyset \neq R, \quad R \text{ is vital} \implies R \text{ is strongly vital–exact w.r.t. } (N, v). \tag{3.9}$$

In order to show our claim, note that by convexity of the core and by the definition of $\mathcal{E}(N, v)$, there exists $x \in C(N, v)$ such that $x(T) > v(T)$ for all $T \in 2^N \setminus \mathcal{E}(N, v)$. Hence, $x(R) = v(R)$ and $x(T) > v(T)$ for all $T \subsetneq R$ with $T \notin \mathcal{E}(N, v)$ so that R is strongly vital–exact.

We proceed by induction on $n = |N|$. If $n = 1$, then N is vital, hence strongly vital–exact, so that the proof is finished in this case. Let the theorem be true for $n \leq t$ and some $t \in \mathbb{N}$ and assume now that $n = t + 1$. If N is vital, then the theorem is true. Hence, we may assume that N is not vital. By Remark 3.1 and Theorem 2.3, there exist a balanced collection $\hat{\mathcal{R}}$ on N and a system $(\hat{\delta}^R)_{R \in \hat{\mathcal{R}}}$ of balancing weights for $\hat{\mathcal{R}}$ such that $N \notin \hat{\mathcal{R}}$ and $\sum_{R \in \hat{\mathcal{R}}} \hat{\delta}^R v(R) = v(N)$. Moreover, for $x \in C(N, v)$, $v(N) = x(N) = \sum_{R \in \hat{\mathcal{R}}} \hat{\delta}^R x(R) = \sum_{R \in \hat{\mathcal{R}}} \hat{\delta}^R v(R)$ so that $R \in \mathcal{E}(N, v)$ for all $R \in \hat{\mathcal{R}}$. As (R, v) is balanced, the inductive hypothesis implies that there exist a balanced collection \mathcal{P}_R on R of strongly vital–exact coalitions w.r.t. (R, v) and a system $(\delta_R^P)_{P \in \mathcal{P}_R}$ of balancing weights for \mathcal{P}_R such that $v(R) = \sum_{P \in \mathcal{P}_R} \delta_R^P v(P)$. Define, for any $P \in \mathcal{P} = \bigcup_{R \in \hat{\mathcal{R}}} \mathcal{P}_R$,

$$\delta^P = \sum_{R \in \{R \in \hat{\mathcal{R}} \mid P \in \mathcal{P}_R\}} \hat{\delta}^R \delta_R^P.$$

We conclude that $\sum_{P \in \mathcal{P}} \delta^P \chi^P = \chi^N$ and $\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N)$. Thus, \mathcal{P} is a balanced collection on N and $\mathcal{P} \subseteq \mathcal{E}(N, v)$ so that the proof is finished by (3.9). \square

Now, the second class of games is constructed as follows. Let (N, v) be a game that satisfies the following property:

$$S \text{ is strongly vital–exact} \implies |S| \leq 2. \tag{3.10}$$

For all $x, y \in X(N, v)$ and all $\alpha \geq 0$ define $z^{\alpha, x, y} \in \mathbb{R}^N$ by

$$z_i^{\alpha, x, y} = \begin{cases} x_i + \min\{y_i - x_i, \alpha\}, & \text{if } y_i \geq x_i, \\ x_i - \min\{x_i - y_i, \alpha\}, & \text{if } x_i \geq y_i, \end{cases} \tag{3.11}$$

and note that $z^{\alpha, x, y}$ is well-defined.

Lemma 3.11. *If (N, v) satisfies (3.10), if $x, y \in C(N, v)$, and if $\alpha \geq 0$, then $z^{\alpha, x, y} \in C(N, v)$.*

Proof. If $\mathcal{C}(N, v) = \emptyset$, then the statement of the lemma is vacuously true. Hence, we assume now that (N, v) is balanced. By Theorem 3.10 there exist a balanced collection \mathcal{P} of strongly vital–exact coalitions on N and a system $(\delta^P)_{P \in \mathcal{P}}$ of balancing weights for \mathcal{P} such that $\sum_{P \in \mathcal{P}} \delta^P v(P) = v(N)$. Let $z = z^{\alpha, x, y}$ and let $i \in N$. If $y_i \geq x_i$, then $z_i \geq x_i \geq v(\{i\})$. If $y_i < x_i$, then $z_i \geq y_i \geq v(\{i\})$. Hence, z is individually rational. Let $P \in \mathcal{P}$. If $|P| = 1$, then $x(P) = y(P) = v(P)$ so that $z(P) = v(P)$. If $|P| = 2$, then $x(P) = y(P) = v(P)$ also implies $z(P) = v(P)$. By (3.10), $z(P) = v(P)$ for all $P \in \mathcal{P}$. We conclude that $z(N) = v(N)$. Now, let $S = \{i, j\}$, $i \neq j$, $i, j \in N$. By (3.10) and Lemma 3.7 it suffices to show that $z(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j \geq x_j$, then $z(S) \geq x(S) \geq v(S)$. If $y_i \geq x_i$ and $y_j < x_j$, then the case $z(S) < y(S)$ may just occur, if $y_i - x_i > \alpha$. However, in this case $z(S) \geq x(S)$. The case $y_i < x_i$ and $y_j \geq x_j$ may be treated similarly. Finally, if $y_i < x_i$ and $y_j < x_j$, then $z(S) \geq y(S)$. Thus, $z \in \mathcal{C}(N, v)$. \square

Proposition 3.12. *If (N, v) satisfies (3.10) and if each $\{i\}$, $i \in N$, is exact, then (N, v) is vital–exact extendable.*

Proof. Let S be a strongly vital–exact coalition and $x \in \mathcal{C}(N, v)$ such that $x(S) = v(S)$. If $|S| = 1$, then the proof is already finished. Hence, we may assume that $S = \{k, \ell\}$ for some $k, \ell \in N$, $k \neq \ell$. Let $y \in \mathcal{C}(N, v)$ such that $y_k = v(\{k\})$ and let $\alpha = x_k - v(\{k\})$. By Lemma 3.11, $z = z^{\alpha, x, y} \in \mathcal{C}(N, v)$. Now, $z_k = y_k = v(\{k\})$ and $z_\ell = \alpha + x_\ell = v(\{k, \ell\}) - v(\{k\})$. By applying the same argument to x and a point $y' \in \mathcal{C}(N, v)$ for which $y'_\ell = v(\{\ell\})$, one may show the existence of $z' \in \mathcal{C}(N, v)$ such that $z'_\ell = v(\{\ell\})$ and $z'_k + z'_\ell = v(\{k, \ell\})$. Notice that the core of (S, v) is the segment

$$\{w \in \mathbb{R}^S \mid w_k \geq v(\{k\}), w_\ell \geq v(\{\ell\}), w_k + w_\ell = v(\{k, \ell\})\}.$$

By convexity of $\mathcal{C}(N, v)$, S is extendable. \square

Proposition 2.1 implies the following result.

Corollary 3.13. *If (N, v) is a balanced game that satisfies (3.10), then the following conditions are equivalent:*

- (1) (N, v) has a stable core.
- (2) (N, v) is vital–exact extendable.
- (3) For each $i \in N$, the singleton $\{i\}$ is an exact coalition.

Note that (3.10) is sharp in the sense that if $|S| \leq 2$ is replaced by $|S| \leq 3$, then Corollary 3.13 is no longer valid. This statement may be shown by means of Example 4.5.

Remark 3.14. Let (N, v) be a balanced game. Schmeidler (1972) presents a simple necessary and sufficient condition for exactness of a singleton $\{i\}$, $i \in N$: The singleton $\{i\}$ is exact if and only if

$$v(\{i\}) = \max \left\{ \sum_{S \subsetneq N} \delta^S v(S) - \delta^N v(N) \mid \delta^T \geq 0 \ \forall T \subseteq N, \sum_{S \subsetneq N} \delta^S \chi^S - \delta^N \chi^N = \chi^{\{i\}} \right\}. \tag{3.12}$$

Note that by Corollary 3.13, (3.12) may be used to check whether a balanced game that satisfies (3.10) has a stable core. Moreover, a similar remark applies to minimum coloring games (see Theorem 4.11).

4. Three remarkable classes of games

Example 3.5 shows that the inverse of Theorem 3.3 does not hold in general. However, each of the three current subsections shows a specific example in which the inverse is true, that is, vital–exact extendability is necessary for core stability. For the examples in Sections 4.2 and 4.3 even vital extendability is a necessary condition for core stability, whereas vital extendability is not necessary for core stability for the class of games that is discussed in Section 4.1. Moreover, in each of the considered classes there are vital–exact extendable games that are not exact extendable.

4.1. Matching games and assignment games

Assignment games (see (4.7) for the definition) as introduced by Shapley and Shubik (1972) are bipartite matching games in the sense of Kern and Paulusma (2003). In order to recall the definition of a matching game, some notation is needed. Let $N \neq \emptyset$ be finite. For any $\emptyset \neq S \subseteq N$ denote $S^2 = \{T \subseteq S \mid |T| \in \{1, 2\}\}$. Any subset of S^2 that is a partition of S is called a *matching* of S . Let $\mathcal{M}(S)$ denote the set of matchings of S . A function $w : N^2 \rightarrow \mathbb{R}_+$ (where \mathbb{R}_+ denotes the set of nonnegative reals) is called a *weight function* on N if

$$w(\{i\}) = 0 \quad \forall i \in N. \tag{4.1}$$

A TU game (N, v) is a *matching game* if there exists a weight function w on N such that

$$v(S) = \max_{\mathcal{P} \in \mathcal{M}(S)} \sum_{T \in \mathcal{P}} w(T) \quad \forall S \subseteq N. \tag{4.2}$$

Remark 4.1. By (4.2) an arbitrary matching game satisfies (3.10) so that Corollary 3.13 is applicable.

Let w be a weight function on N and let $v = v^w$ be defined by (4.2). We say that (N, v^w) is the matching game defined by w . Moreover, a matching \mathcal{P} of N is optimal for w if $v^w(N) = \sum_{S \in \mathcal{P}} w(S)$. For $i \in N$ let $w_i : N^2 \rightarrow \mathbb{R}_+$ be defined by

$$w_i(\{k\}) = 0 \quad \text{and} \quad w_i(\{k, \ell\}) = (w(\{k, \ell\}) - w(\{i, k\}) - w(\{i, \ell\}))_+ \quad \forall k, \ell \in N, k \neq \ell,$$

where $t_+ = \max\{t, 0\}$ for $t \in \mathbb{R}$, and note that w_i is a weight function.

Theorem 4.2. If (N, v) is a matching game defined by a weight function w on N , then the following conditions are equivalent:

- (1) (N, v) has a stable core.
- (2) For all $i \in N$,

$$\mathcal{C}(N, v^{w_i}) \neq \emptyset \quad \text{and} \quad v^{w_i}(N) = v(N) - \sum_{k \in N} w(\{i, k\}). \tag{4.3}$$

- (3) Each singleton is exact.

Proof. Let $i \in N$. Assume that (N, v) has a stable core. By Proposition 2.1 there exists $x \in \mathcal{C}(N, v)$ such that $x_i = 0$. Hence, $x_j \geq w(\{i, j\})$ for all $j \in N$. Let $y \in \mathbb{R}^N$ be defined by $y_j = x_j - w(\{i, j\})$ for all $j \in N$. Then $y_j \geq 0$ for all $j \in N$ and, for all $j, k \in N, k \neq j$,

$$y_j + y_k = x_j + x_k - w(\{i, j\}) - w(\{i, k\}) \geq (w(\{j, k\}) - w(\{i, j\}) - w(\{i, k\}))_+ = w_i(\{j, k\}).$$

Moreover, $y(N) = v(N) - \sum_{k \in N} w(\{i, k\})$ so that (4.3) has been verified.

Now assume that (4.3) is satisfied. Let $y \in \mathcal{C}(N, v^{w_i})$ and define $x \in \mathbb{R}^N$ by $x_k = y_k + w(\{i, k\})$ for all $k \in N$. Then $x(S) \geq w(S)$ for all $S \in \mathcal{P}$ and $x(N) = v(N)$ so that $x \in \mathcal{C}(N, v)$. As $w_i(\{i, k\}) = 0$ by definition for all $k \in N$, $v^{w_i}(S \cup \{i\}) = v^{w_i}(S)$ for all $S \subseteq N$ so that $y_i = 0$. We conclude that $x_i = 0$. By Remark 4.1, Corollary 3.13 completes the proof. \square

Corollary 4.3. If (N, v) is a matching game with a stable core defined by a weight function w on N and if \mathcal{P} is an optimal matching for w , then, for $i, i', j \in N$ with $i \neq j, \{i'\}, \{i, j\} \in \mathcal{P}$,

$$w(\{i', k\}) = 0 \quad \forall k \in N \setminus \{i'\}; \tag{4.4}$$

$$w(\{i, j\}) \geq w(\{i, k\}) + w(\{j, k\}) \quad \forall k \in N \setminus \{i, j\}. \tag{4.5}$$

Proof. Let $k \in N$. By (4.3),

$$\begin{aligned} v^{w_k}(N) &\geq \sum_{S \in \mathcal{P}} w_k(S) = \sum_{S \in \mathcal{P}} \left(w(S) - \sum_{\ell \in S} w(\{k, \ell\}) \right)_+ \geq \sum_{S \in \mathcal{P}} \left(w(S) - \sum_{\ell \in S} w(\{k, \ell\}) \right) \\ &= \sum_{S \in \mathcal{P}} w(S) - \sum_{\ell \in N} w(\{k, \ell\}) = v(N) - \sum_{\ell \in N} w(\{k, \ell\}) = v^{w_k}(N), \end{aligned}$$

so that

$$w(S) - \sum_{\ell \in S} w(\{k, \ell\}) \geq 0 \quad \forall S \in \mathcal{P}, k \in N. \tag{4.6}$$

If $\{i'\} \in \mathcal{P}$, then (4.6) applied to $S = \{i'\}$ yields $0 = w(\{i'\}) \geq w(\{i', k\}) \geq 0$ (see (4.1)) so that (4.4) is shown. If $\{i, j\} \in \mathcal{P}$ with $i \neq j$, then (4.6) applied to $S = \{i, j\}$ shows (4.5). \square

A graph is a pair $G = (V, E)$, where V is a finite nonempty set, called the set of vertices, and E is a set of 2-element subsets of V , called the set of edges. Note that the weight function w defines the graph (N, E^w) by $S \in E^w$ if $w(S) > 0$.

We assume now that $|N| \geq 2$. The matching game (N, v^w) is an assignment game if (N, E^w) is bipartite, that is, there is a partition $\{P, Q\}$ of N such that

$$|S \cap P| = |S \cap Q| = 1 \quad \forall S \in E^w. \tag{4.7}$$

Let (N, v^w) be an assignment game, let $\{P, Q\}$ be a partition of N that satisfies (4.7), and denote $v = v^w$. Thus, the nonnegative real matrix $A = (w(\{i, j\}))_{i \in P, j \in Q}$, the assignment matrix of (N, v) , determines the assignment game (N, v) .

The theory developed so far enables us to reprove Theorem 1 of Solymosi and Raghavan (2001): Assume without loss of generality that $|P| \leq |Q|$. Then there is an injective mapping $b : P \rightarrow Q$ such that $\{\{i, b(i)\} \mid i \in P\} \cup \{\{j\} \mid j \in Q \setminus b(P)\}$ is an optimal matching of N for w , that is, $v(N) = \sum_{i \in P} w(\{i, b(i)\})$. Now, the aforementioned theorem may be formulated as follows. The assignment game (N, v) has a stable core if and only if

$$w(\{i, j\}) = 0 \quad \forall j \in Q \setminus b(P), \quad \forall i \in P; \tag{4.8}$$

$$w(\{i, b(i)\}) = \max_{r \in Q} w(\{i, r\}) = \max_{r \in P} w(\{r, b(i)\}) \quad \forall i \in P. \tag{4.9}$$

A careful inspection of Corollary 4.3 shows that (4.8) and (4.9) coincide with (4.4) and (4.5), respectively. In order to verify the if direction, it should be noted that, if w is a weight function such that (N, E^w) is bipartite, then for $i \in N$, (N, E^{w_i}) is bipartite. As assignment games are balanced (see Shapley and Shubik, 1972), the first part of (4.3) is automatically satisfied. Moreover, a careful inspection of the definition of w_i shows that the second part of (4.3) is implied by (4.8) and (4.9).

The following example shows that neither exact extendability nor vital extendability is necessary for core stability for assignment games.

Example 4.4. Let

$$A = \begin{pmatrix} 6 & 4 & 0 \\ 0 & 6 & 0 \\ 4 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix},$$

and let (N, v_4) be the assignment game defined by A , where 1, 2, and 3 are the “row” players and 4, 5, and 6 are the “column” players. The unique optimal matching $\{\{i, b(i)\} \mid i = 1, 2, 3\}$ is given by $b(i) = 3 + i$ for $i = 1, 2, 3$. Hence, (4.8) and (4.9) are satisfied so that (N, v_4) has a stable core. Moreover, $x = (3, 5, 1, 3, 1, 5) \in \mathcal{C}(N, v_4)$ and $x(S) = v_4(S)$, where $S = \{1, 3, 4, 5\}$. Now, S is not extendable, because $(4, 0, 4, 0) \in \mathcal{C}(S, v_4)$ and any $y \in \mathcal{C}(N, v_4)$ must assign $w(\{i, b(i)\})$ to any coalition $\{i, b(i)\}$ of optimally matched players, e.g., satisfies $y_1 + y_4 = w(\{1, 4\}) = 6$. We conclude that (N, v_4) is not exact extendable. In order to show that (N, v_4) is vital extendable, it suffices to show that $\{1, 5\}$ and $\{3, 4\}$ are extendable. A careful inspection of the core elements $(0, 2, 0, 6, 4, 6)$, $(4, 6, 6, 2, 0, 0)$, $(6, 6, 4, 0, 0, 2)$, $(2, 0, 0, 4, 6, 6)$ shows that the aforementioned coalitions are extendable. Moreover, we remark that there are also assignment games with a stable core that are not vital extendable. Indeed, let (N, v_5) be the assignment game defined by B . As each pair (i, j) , $i \in P$, $j \in Q$, belongs to an optimal matching except the pair $(3, 4)$, we conclude that $\mathcal{C}(N, v_5) = \{(\alpha, \alpha, \alpha, 2 - \alpha, 2 - \alpha, 2 - \alpha) \mid 0 \leq \alpha \leq 2\}$. Consequently, the vital coalition $\{3, 4\}$ is not exact and, hence not extendable.

4.2. Simple flow games

Kalai and Zemel (1982) present two equivalent representations of totally balanced games: A game is totally balanced game if and only if (a) it is a flow game or (b) it is the minimum of finitely many additive games. The following example shows that even for the minimum of two additive games, the simplest nontrivial case in (b), vital-exact extendability may not be necessary for core stability. Moreover, for “simple” flow games we shall derive that vital extendability is necessary and sufficient for core stability.

Example 4.5. Let $N = \{1, \dots, 6\}$, let $\lambda = (2, 1, 1, 2, 1, 1)$, let $N^1 = \{1, 2, 3\}$, let $N^2 = \{4, 5, 6\}$ and let (N, v) be the game given by $v(S) = \min_{i=1,2} \lambda(S \cap N^i)$. The game (N, v) is exact (see, e.g., Raghavan and Sudhölter, 2005) and it has a stable core. Indeed, $\mathcal{C}(N, v)$ is the convex hull of $(2, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 2, 1, 1)$ (see, e.g., Rosenmüller, 2000) so that if $y \in I(N, v) \setminus \mathcal{C}(N, v)$ is not dominated by any core element via some 2-person coalition, then y satisfies $y_2 = y_3$ and $y_5 = y_6$. Hence, y is dominated by some core element via $S = \{1, 5, 6\}$ or via $T = \{2, 3, 4\}$ and it also follows that S and T are strongly vital-exact coalitions. These coalition are not extendable, because for any $x \in \mathcal{C}(N, v)$, $x_2 = x_3$ and $x_5 = x_6$, but $\mathcal{C}(S, v)$ contains some z with $z_5 \neq z_6$ (e.g., given by $z_1 = z_6 = 1/2, z_5 = 1$) and a similar statement holds for T .

Adopting the notation of Sun and Fang (2007), who characterized the simple flow games that have a stable core, $D = (V, E, s, t)$ is a simple (directed) network with source s and sink t , if V is the vertex set, $E \neq \emptyset$ is the arc set, and s and t are distinct vertices in V . The term “simple” refers to the fact that all arcs have the same capacity, let us say 1. The flow game (E, v^D) associated with $D = (V, E, s, t)$ is the TU game defined by the requirement that, for any $\emptyset \neq S \subseteq E$, $v^D(S)$ is the maximal flow from s to t in the network (V, S, s, t) . A game (N, v) is a simple flow game if it is the game associated with some simple directed network with a source and a sink.

A (simple) path in a network $D = (V, E, s, t)$ is a sequence of arcs from s to t that visits each vertex at most once. It is well known that

$$v^D(S) \text{ is the maximal number of arc-disjoint paths in } (V, S, s, t) \text{ for all } S \in 2^E \setminus \{\emptyset\}. \tag{4.10}$$

Let $D = (V, E, s, t)$ be a simple network with source and sink and denote $v = v^D$.

Remark 4.6. If a coalition S is vital and $v(S) > 0$, then $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneq S$. Indeed, by (4.10), the elements of S , suitably ordered, must form a path.

An arc $e \in E$ is called a *dummy arc*⁵ if there exists a path containing e and if $v(E \setminus \{e\}) = v(E)$. We recall that a *cut* of D is a coalition $C \subseteq E$ such that each path contains an arc of C . For a proof of the following “max-flow min-cut” theorem see, e.g., Ford and Fulkerson (1962):

$$v^D(E) = \min\{|C| \mid C \text{ is a cut of } D\}. \tag{4.11}$$

We are now able to recall Theorem 3 of Sun and Fang (2007).

Theorem 4.7. Let $D = (V, E, s, t)$ be a simple network with source and sink. Then (E, v^D) has a stable core if and only if E does not contain any dummy arc.

We use the preceding theorem and the following lemma and remark to show that vital extendability is necessary for core stability in the case of simple flow games. Let $D = (V, E, s, t)$ be a simple network with source and sink.

Lemma 4.8. If E does not contain any dummy arc and if $e \in E$ satisfies $v^D(E \setminus \{e\}) < v^D(E)$, then there exists a minimum cut C with $e \in C$.

Proof. By (4.10) there are $v^D(E)$ arc-disjoint paths. We may assume that $v^D(E) > 1$. As $v^D(E) > v^D(E \setminus \{e\})$, the arc e must be contained in one of the paths and $v^D(E \setminus \{e\}) = v^D(E) - 1$. Hence, if C' is a minimum cut of $(V, E \setminus \{e\}, s, t)$, then $C' \cup \{e\}$ is a minimum cut of D by (4.11). \square

Remark 4.9. In a constructive way Kalai and Zemel (1982, p. 478) show that the core of an arbitrary flow game is nonempty. Applied to a simple flow game (N, v) associated with the simple network $D = (V, E, s, t)$ they prove that, for any minimum cut C of D , $\chi^C \in C(E, v^D)$.

Proposition 4.10. A simple flow game (N, v) has a stable core if and only if it is vital extendable.

Proof. Let $D = (V, E, s, t)$ be a simple network with source and sink and let (E, v) be the associated simple flow game. As the if direction is valid by Theorem 3.3, we assume now that (E, v) has a stable core. Let S be a vital coalition. If $v(S) = 0$, then $|S| = 1$ and, by Proposition 2.1, S is extendable. If $v(S) > 0$, then, by Remark 4.6, $v(S) = 1$ and $v(T) = 0$ for all $T \subsetneq S$ and the elements of S form a path. By Lemma 4.8 and Remark 4.9, for any $e \in S$, there exists $x \in C(E, v)$ such that $x_e = 1$ and $x_{e'} = 0$ for all $e' \in S \setminus \{e\}$. However, $C(S, v)$ is the convex hull of those core elements when restricted to S . \square

It should be remarked that Fang et al. (2007, p. 444) present an example of a simple flow game (associated with G_3) that has a stable core and is not extendable. (Indeed, the 4-person coalition corresponding to the arcs that are marked by + is not exact, but the core of the corresponding subgame is nonempty.)

4.3. Minimum coloring games

Deng et al. (1999) introduced minimum coloring games and we basically adopt the notation of Bietenhader and Okamoto (2006). Let $G = (V, E)$ be a graph (see Section 4.1). For any $U \subseteq V, U \neq \emptyset$, let G^U denote the subgraph of G whose vertex set is U and whose edges are those edges in E that are subsets of U .

The graph G is *complete* if E is the set of all 2-element subsets of V . A nonempty set $U \subseteq V$ is a *clique* if G^U is complete. Let $\omega(G)$ denote the size of a maximum clique. A *coloring* of G is a mapping $c : V \rightarrow \mathbb{R}$ satisfying $c(i) \neq c(j)$ for all $\{i, j\} \in E$. A *minimal coloring* is a coloring c such that $|c(V)|$ is minimal. Let $\gamma(G)$ denote the *chromatic number* of G , i.e., $\gamma(G) = |c(V)|$ for any minimal coloring of G . A set $U \subseteq V, U \neq \emptyset$, is *independent* if $\gamma(G^U) = 1$. The graph G is *perfect* if $\omega(G^U) = \gamma(G^U)$ for all $U \in 2^V \setminus \{\emptyset\}$.

Let $G = (V, E)$ be a graph. The *minimum coloring game* on G is the TU game (N, v^G) defined by the following requirements: (1) $N = V$; (2) $v^G(S) = |S| - \gamma(G^S)$ for⁶ all $S \in 2^V \setminus \{\emptyset\}$.

Theorem 4.11. Let (N, v) be a balanced minimum coloring game. Then the following conditions are equivalent:

⁵ Sun and Fang (2007) use this term although a dummy arc is not a *dummy player*. Indeed, an arc is a dummy player if and only if it either connects s and t or it is not contained in any path.

⁶ Bietenhader and Okamoto (2006) consider the “cost” game whose coalition function simply assigns $\gamma(G^S)$ to any coalition S . We consider the “cost sharing” game instead so that, e.g., the definition of the core remains unchanged.

- (1) (N, v) has a stable core.
- (2) (N, v) is vital extendable.
- (3) Every singleton is exact w.r.t. (N, v) .

We postpone the proof of Theorem 4.11 and first prove the following lemma.

Lemma 4.12. *Let (N, v) be a minimum coloring game on the graph $G = (V, E)$. Then $\emptyset \neq S \subseteq N$ is vital if and only if S is independent.*

Proof. If S is independent, then $v(T) = |T| - 1$ for all $\emptyset \neq T \subseteq S$. Let $x \in \mathbb{R}^S$ be defined by $x_i = \frac{|S|-1}{|S|}$ for all $i \in S$. Then $x(S) = v(S)$ and $x(T) > v(T)$ for all $\emptyset \neq T \subsetneq S$ so that S is vital. Conversely, assume now that S is a coalition with $v(S) < |S| - 1$. It remains to show that S is not vital. Let $c : S \rightarrow \mathbb{R}$ be a minimal coloring of G^S and let $i \in S$. Then $T = \{j \in S \mid c(j) \neq c(i)\} \neq \emptyset$ and c_T (the restriction of c to T) is a minimal coloring of G^T . We conclude that $v(S) = v(T) + v(S \setminus T)$ and, hence, that S is not vital. \square

Proof of Theorem 4.11. By Proposition 2.1, Theorem 3.3 and (3.3) it remains to show that (3) implies (2). Let S be a vital coalition and $y \in \mathcal{C}(N, v)$. For any $j \in N$, $v(N) - v(N \setminus \{j\}) + y(N \setminus \{j\}) \geq v(N) = y_j + y(N \setminus \{j\})$. We conclude that $y_j \leq v(N) - v(N \setminus \{j\})$. As $v(N) - v(N \setminus \{j\}) \leq 1$ for any minimum coloring game, we conclude that $y_j \leq 1$. Now, let $i \in S$. By (3), there exists $x \in \mathcal{C}(N, v)$ with $x_i = v(\{i\}) = 0$. By Lemma 4.12, $v(S) = |S| - 1$. Therefore, $x_j = 1$ for all $j \in S \setminus \{i\}$ and convexity of the core completes the proof. \square

We now use Theorem 4.11 to characterize minimum coloring games that have stable cores.

Theorem 4.13. *Let $G = (N, E)$ be a graph, let c be a minimal coloring of G , and denote, for $k \in c(N)$, $T_k = \{i \in N \mid c(i) \neq k\}$. The minimum coloring game (N, v^G) has a stable core if and only if for any $k \in c(N)$,*

$$\gamma(G^{T_k \cup \{i\}}) = \gamma(G) \quad \text{and} \quad (T_k \cup \{i\}, v^{G^{T_k \cup \{i\}}}) \text{ is balanced } \forall i \in N \setminus T_k. \tag{4.12}$$

Proof. Let $v = v^G$. If $\gamma(G) = 1$, then (N, v) has a stable core and the proof is immediate. Hence, we may assume that $\gamma(G) \geq 2$ so that for any $k \in c(N)$, $N \setminus T_k$ is independent and

$$v(T_k) + v(N \setminus T_k) = |T_k| - (\gamma(G) - 1) + |N \setminus T_k| - 1 = |N| - \gamma(G) = v(N). \tag{4.13}$$

In order to verify the only if direction let $k \in c(N)$ and $i \in N \setminus T_k$. By Theorem 4.11 there exists $x \in \mathcal{C}(N, v)$ such that $x_i = 0$. By (4.13), $x(N \setminus T_k) = v(N \setminus T_k)$ and $x(T_k) = v(T_k)$. As $v(T_k) = x(T_k) = x(T_k \cup \{i\}) \geq v(T_k \cup \{i\}) \geq v(T_k)$, $\gamma(G^{T_k \cup \{i\}}) = \gamma(G)$ and $x_{T_k \cup \{i\}} \in \mathcal{C}(T_k \cup \{i\}, v)$.

In order to verify the if direction, let $i \in N$. By Theorem 4.11 it suffices to show that there exists $x \in \mathcal{C}(N, v)$, $x_i = 0$. Let $k = c(i)$. By (4.12), $v(T_k \cup \{i\}) = v(T_k)$ and there exists $y \in \mathcal{C}(T_k \cup \{i\}, v)$. As $v(T_k \cup \{i\}) = v(T_k) + 0 = v(T_k) + v(\{i\})$, we conclude that $y(T_k) = v(T_k)$ and $y_i = 0$. Let $x \in \mathbb{R}^N$ be given by $x_{T_k \cup \{i\}} = y$ and $x_j = 1$ for all $j \in N \setminus (T_k \cup \{i\})$. By (4.13), $x(N) = v(N)$ so that $x \in \mathcal{C}(N, v)$. \square

It should be noted that a minimum coloring game of a graph is balanced if there exists a coloring such that (4.12) is valid for some k and some i . The foregoing theorem generalizes the following result of Bietenhader and Okamoto (2006, Theorem 4.1).

Corollary 4.14. *The minimum coloring game on a perfect graph G has a stable core if and only if every vertex of G belongs to a maximum clique of G .*

Proof. We may assume that $\gamma(G) > 1$. Let $G = (N, E)$ be a perfect graph, let c be a minimal coloring of G , let $v = v^G$, let $i \in N$, let $k = c(i)$, and let $T = \{j \in N \mid c(j) \neq k\}$. If (N, v) has a stable core, then, by (4.12), $\omega(G^T) = \gamma(G^T) = \gamma(G) - 1$ and $\omega(G^{T \cup \{i\}}) = \gamma(G^{T \cup \{i\}}) = \gamma(G) = \omega(G)$ so that i is in a maximum clique. If, on the other hand, i is in some maximum clique, then $\gamma(G) = \omega(G^{T \cup \{i\}})$ so that (4.12) is satisfied for $T = T_k$ and i . \square

Note that Bietenhader and Okamoto (2006, p. 424) present a perfect graph (G_1) with some vertex that does not belong to a maximal clique, that is, the first condition of (4.12) is violated for some T_k and some i , so that the resulting minimum coloring game does not have a stable core.

Example 4.15. Let G_2 be the perfect graph that consists of two disjoint triangles that are connected via one edge and may be found in Bietenhader and Okamoto (2006, p. 424). (For a characterization of extendable minimum coloring games on perfect graphs see their Theorem 4.2.) So, $G_2 = (N, E)$, where

$$N = \{1, \dots, 6\} \quad \text{and} \quad E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 4\}\}.$$

Let $v = v^{G^2}$ and $T = \{1, 4\}$. Then $x(T) \geq 1$ for any $x \in \mathcal{C}(N, v)$. Note that $(0, 0, 0, 1, 1, 1) \in \mathcal{C}(N, v)$ so that $S = \{1, 2, 4\}$ is exact. As $(0, 1, 0) \in \mathcal{C}(S, v)$, S is exact and not extendable. By Corollary 4.14, (N, v) has a stable core.

We present a graph that has a minimal coloring so that exclusively the second condition of (4.12) is violated for a unique T_k and a unique i . Moreover, we present a minimum coloring game on a nonperfect graph that has a stable core.

Example 4.16. Let $G = (N, E)$ be defined by $N = \{1, \dots, 7\}$ and

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\} \cup \{\{1, 6\}, \{5, 6\}, \{3, 7\}, \{4, 7\}\}.$$

Let $v = v^G$, let $S = \{1, \dots, 5\}$, and note that G^S is a pentagon so that $\gamma(G^S) = 3 > 2 = \omega(G^S)$. We conclude that G is not perfect and it is well known that $\mathcal{C}(S, v) = \emptyset$. Moreover, let $c : N \rightarrow \mathbb{R}$ be defined by $c(1) = c(4) = 1$, $c(3) = c(5) = 2$, and $c(2) = c(6) = c(7) = 3$. Thus, c is a minimal coloring of G and, with the notation of Theorem 4.13,

$$T_1 = N \setminus \{1, 4\}, \quad T_2 = N \setminus \{3, 5\}, \quad T_3 = N \setminus \{2, 6, 7\}.$$

As $S_1 = \{1, 5, 6\}$ and $S_2 = \{3, 4, 7\}$ are cliques, we conclude that $\omega(G^{T_k \cup \{i\}}) = \gamma(G^{T_k \cup \{i\}}) = 3 = \gamma(G)$ and, hence, $\mathcal{C}(T_k \cup \{i\}, v) \neq \emptyset$ for $k = 1, 2$ and all $i \in N \setminus T_k$. Similarly, if $i \in \{6, 7\}$, then $\omega(G^{T_3 \cup \{i\}}) = \gamma(G^{T_3 \cup \{i\}}) = 3 = \gamma(G)$ and $\mathcal{C}(T_3 \cup \{i\}, v) \neq \emptyset$. Finally, $T_3 \cup \{2\} = \{1, \dots, 5\}$ so that $\gamma(G^{T_3 \cup \{2\}}) = 3 = \gamma(G)$ even in this case. Thus, $\mathcal{C}(T_3 \cup \{2\}, v) = \emptyset$ so that there is a unique balancedness condition that is violated in (4.12).

If $G' = (N, E')$ is the graph that differs from G only inasmuch as E' contains the additional edge $\{2, 7\}$, then we may define a minimal coloring $c' : N \rightarrow \mathbb{R}$ by $c'(1) = c'(7) = 1$, $c'(3) = c'(5) = 2$, and $c'(2) = c'(4) = c'(6) = 3$. Now there is the additional maximal clique $S_3 = \{2, 3, 7\}$ so that (4.12) is valid. In fact, $\chi^{S_i} \in \mathcal{C}(N, v)$ for $i = 1, 2, 3$ so that any singleton is exact and Theorem 4.11 may be applied directly to show that the minimum coloring game on the nonperfect graph G' has a stable core.

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