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PROBABILISTIC BOUNDS ON THE VIRTUAL MULTIPLIERS
IN DATA ENVELOPMENT ANALYSIS

POLYHEDRAL CONE CONSTRAINTS

Original October 1996
Revised July 1997
Revised November 1997
Submitted November 1997 to Journal of Productivity Analysis
Revised July 1998
Keywords: DEA, assurance regions, chance constrained programming, stochastic DEA

Abstract.

The paper is concerned with the incorporation of polyhedral cone constraints on the virtual multipliers in DEA. The incorporation of probabilistic bounds on the virtual multipliers based upon a stochastic benchmark vector is demonstrated. The suggested approach involves a stochastic (chance constrained) programming model with multipliers constrained to the cone spanned by confidence intervals for the components of the stochastic benchmark vector at varying probability levels. Consider a polyhedral assurance region based upon bounded pairwise ratios between multipliers. It is shown that in general it is never possible to identify a "center-vector", defined as a vector in the interior of the cone with identical angles to all extreme rays spanning the cone. Smooth cones are suggested if an asymmetric variation in the set of feasible relative prices is to be avoided.

1. Introduction and motivation.

DEA as developed in the seminal paper by Charnes et al. (1978,1979) and generalized by Banker et al. (1984) is often characterized as a methodology based upon very few maintained hypotheses and an often stated advantage is that no value information is needed. As formulated by Thompson et al. (1990) "DEA .. does not require any a priori weights in a frontier analysis of the inputs and outputs. DEA is value-free, which is a strength and a weakness". DEA only requires a distinction between inputs and outputs along with a number of additional maintained hypotheses related to e.g. disposability characteristics, returns to scale characteristics and convexity of various sets.

Why is the non-requirement of a priori weights sometimes considered a weakness and which are the reasons for introduction of value information in DEA? At least 3 different but related arguments can be given:

i) Cooper et al. (1996) note that allocative efficiency can be of limited value in practical applications because of data requirements and unjustifiable assumptions. Exact information on prices is difficult to obtain and prices can be subject to variation in short periods. A specification of admissible relative prices by introduction of bounded pairwise price ratio constraints as in the so-called AR approach suggested by Thompson et al. (1986 & 90) is one way to proceed. The inclusion of value information can in this case be seen as a reorientation of the DEA model from pure technical efficiency measurement towards allocative efficiency measurement. Thompson et al. (1990) summarize the AR principle by noting that: "AR principles were developed and used to bound the virtual multipliers .. to work from technical efficiency towards overall efficiency"
ii) The weakness caused by DEA being a value-free methodology can be argued to occur e.g. in model specifications with many inputs and outputs (compared to the number of DMUs). Many DMUs may in this situation operate with a mix of inputs and outputs that is specialized in the sense that dominance is not possible. Thus, the information provided by the classification of a DMU as efficient tends to become less useful when the number of inputs and/or outputs is increased. An analysis characterizing most or all DMUs as efficient may for this reason be of limited value as a foundation for decision making. The analyst may omit or aggregate an appropriate number of inputs and/or outputs from the analysis in order to avoid the problem. However, omitting outputs which have required the use of inputs for their production implies bias and an aggregation in the form of a weighted sum requires exact information on relative prices. In many applications such knowledge is often difficult or impossible to get. In addition, prices often vary which introduces additional choices and assumptions into the analysis.

One way to avoid this problem is the introduction of ARs as the above mentioned bounded pairwise price ratio constraints. Introducing such bounds on the dual variables, e.g. in the CCR-model, can be given the interpretation of a "soft" aggregation of some/all inputs and some/all outputs. If we tighten the assurance region by letting the upper and lower bounds get close to each other for all relative prices then we approach a DEA analysis with only one aggregated input and only one aggregated output. Thompson et al. (1990) note that the "The AR approach allows one to augment successively an AR until a satisfactory level of efficiency refinement is achieved in the 'eyes of user', who is the ultimate 'consumer' of the modeling product".

iii) The DEA-estimated production possibility set is in some applications simply too small. If information on feasible rates of substitution between different pairs of outputs or inputs is available then this information can be used to enlarge the production possibility set by the inclusion of a suitable set of ARs. Assume that characteristics of globally feasible rates of substitutions in the input output space are known. This information may be available in the form of either lower and upper bounds on the rates of substitution or by their admissible deviations from a benchmark vector.

Observe that we argue for an expansion of the set of possible input output combinations beyond the traditional envelopment-estimated production possibility set in relation to iii). Hence, an AR based on iii) resulting in lower input oriented DEA efficiency scores reflects the existence of dominating input output combinations within the possibility set with lower input values for a given output. A lower efficiency score reflects in this case a decrease in the pure technical efficiency. On the other hand, the inclusion of an AR based on i) also resulting in lower input oriented DEA efficiency scores does not reflect the existence of a dominating input output combination within the possibility set with lower input values for a given output. A lower efficiency score reflects in this case that the overall efficiency for a particular DMU is lower than its pure technical efficiency score.

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1 The "envelopment estimated production possibility set" is in the CCR-model (Charnes et al. (1978,1979)) the cone generated by all positive linear cobinations of vectors from the convex hull of all DMUs set added to $\mathbb{R}^n_+ \times \mathbb{R}^s_-$ ($n$ inputs and $s$ outputs).
The efficiency analysis related to the Danish hospital sector in Olesen and Petersen (1998) is an example of how information on feasible rates of substitution between different pairs of outputs can be used to enlarge the production possibility set. In this analysis a 484 dimensional input output space is used to evaluate the relative efficiency of 70 Danish hospitals. The 483 virtual output multipliers are constrained by ARs based on distributional characteristics of measures of the average resource consumption in 483 patient categories (based partly on the system of Diagnosis Related Groups). For many subgroups of the services that constitute a treatment in each of these patient categories we can argue, that input is homogeneous and can be reallocated to other activities based upon measures of the average resource consumption. Hence, we can argue, that for each pair of patient categories we may use the ratio of the corresponding average resource consumption as a measure of a possible rate of substitution. Introducing confidence regions in the analysis makes it possible to use the measures of the relative precision of the use of resources to specify a range of possible rates of substitution. This range is specified in the form of an AR along the lines presented in the next section.

A flexible and often used approach for specifying an AR in the output multiplier space is as follows (see Thompson et al. (1986)): $^2$

$$
\mathbf{u} \in C_1(a_{i,j}, b_{i,j}, (i, j) \in IJ) \equiv \left\{ \mathbf{u} \mid a_{i,j} \leq \frac{u_i}{u_j} \leq b_{i,j}, (i, j) \in IJ \right\}
$$

$\mathbf{u} = (u_1, \ldots, u_s)^T$ is the vector of output multipliers in an output space of dimension $s$ and $IJ \equiv \{(i, j) : i = 1, \ldots, s, j = i + 1, \ldots, s\}$ for notational convenience. Clearly, (1) defines a polyhedral cone $C_1()$.

(1) involves an a priori determination of the set of parameters $(a_{i,j}, b_{i,j}), (i, j) \in IJ$, defining lower and upper bounds for the respective price ratios. The use of an expert opinion or historical data are common approaches for an identification of these parameters.$^3$ Expert opinions or judgments can in some cases be made with high precision but may be subject to significant uncertainty. The incorporation of uncertainty in the determination of the bounds involved in (1) in a DEA model with ARs is a main issue to be addressed in this paper. It is shown that it is possible to incorporate information concerning the uncertainty in a determination of the set of parameters $(a_{i,j}, b_{i,j}), (i, j) \in IJ$, into the analysis, if this uncertainty can be quantified in the form of distributional characteristics.

$^2$ Focus in this paper is on ARs in the output multiplier space. ARs in the input multiplier space are derived similary and the results to be derived for ARs in the output multiplier space extend easily to the input multiplier space.

$^3$ Some examples of DEA applications with ARs like (1), related to the probabilistic approach proposed here are as follows: Thompson et al. (1997) specify flexible ARs (the median plus or minus one quartile of the respective price/cost data). Schaffnit et al. (1997) construct $s(s - 1)/2$ output multiplier constraints from the mean value plus/minus a deviation. Athanassopoulos (1997) uses as upper and lower bounds the 25% upper and lower quartile score obtained from the distribution of tradeoff ratios across all branches. Taylor et al. (1997) specify bounds obtained from the range of nominal interest rates for the loan and deposit portfolios of thirteen banks. Banker and Morey (1989) specify two intervals for cost of a recruiter man-year and the cost of the AFC award. These two ranges could be the domains for two cost distributions.
It is in some applications possible to incorporate uncertainty on the parameters in an AR by specification of the distributional characteristics for a stochastic norm- or benchmark-vector. It is shown in Section 2 that the polyhedral AR defined by (1) can be used to impose the requirement that the set of feasible relative dual prices must be restricted to sets of realizations of all corresponding relative prices for the stochastic benchmark-vector considered of high likelihood. The lower and upper bounds in (1) will be determined as a function of the left and right end-points in confidence intervals for each component in the stochastic benchmark-vector. It may in other applications be considered more appropriate to specify the lower and/or upper bound for each pair of relative multipliers as two stochastic one-sided target-vectors. The identification of upper and lower bounds in (1) as a function of the left and right end point in one-sided confidence intervals for the components in the two stochastic target-vectors is addressed in Section 3.

The probabilistic ARs suggested in Sections 2 and 3 can be specified as (1) thus demonstrating the flexibility of this specification. Hence, the probabilistic ARs are polyhedral cones defined by the bounds imposed on pairwise ratios between multipliers as in the standard approach. The distinguishing feature of the suggested approach is that the uncertainty involved in an expert evaluation or by an empirical distribution based on historical data of the parameters in (1) is used for the specification of bounds in the AR.

In Section 4 it is shown that with a polyhedral cone like \( C_1() \) in (1) it is in general never possible to identify a "center-vector", defined as a vector in the interior of the cone with identical angles to all extreme rays spanning the cone. The problem is of importance both in the context of the present study and in a standard AR formulation since the suggested probabilistic cone constraints can be specified as (1). An asymmetric variation in the set of feasible relative prices may well be appropriate in some applications, but can also be seen as an undesirable characteristic of an AR-specification by (1). It is shown in the Appendix that the problem can

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4 The stochastic benchmark vector as starting point for imposing probabilistic assurance region originates from the above mentioned efficiency analysis of the Danish hospital sector (Olesen and Petersen (1998)). For each DRG we have access to an estimate of the average cost of a discharge and to the distribution of actual cost (or more precisely the distribution of actual length of stay). The distribution of cost in each DRG constitutes a stochastic benchmark vector in multiplier space. Hence, incorporating the precision of each relative average cost into the analysis was done by spanning cones on confidence regions for each pair of distributions of actual cost for each pair of DRGs. Clearly, the length of stay distribution is typically very skew and poorly approximated by a normal distribution. Hence, to construct the confidence interval for each DRG, we used a flexible approximation, namely the truncated normal distribution with endogenous truncation point. Next, we performed a maximum likelihood estimation of the mean, the variance and the truncation point for each DRG and finally estimated the upper and lower value of cost corresponding to a two sided confidence interval on an \( \alpha \) percent level.
be circumvented by specifying the ARs as smooth cones. Section 5 provides some concluding remarks.

The AR approach in DEA considered in this paper is designed in order to eliminate a set of relative output virtual multipliers (and/or a set of relative input multipliers), which for some reason is considered unacceptable. The approach involves the additional inclusion of a set of constraints defining a cone in the output and/or in the input multiplier space. Focus in this paper is on the specification of probabilistic ARs exclusively with no linked cone constraints. Hence, no hypotheses concerning returns to scale are maintained since the concept of probabilistic ARs as suggested in this paper applies no matter such maintained hypothesis.

2. Polyhedral assurance cones generated from a priori information in the form of a stochastic norm vector.

Assume that we have a stochastic "norm-vector" \( \mathbf{e} = [e_1, \ldots, e_s]^T \), where each component \( e_i, i = 1, \ldots, s \) is a random variable and these \( s \) random variables are distributed according to some multivariate distribution. Consider some set of realizations of the random vector \( \mathbf{e} \) that is regarded as having a high likelihood (for example a confidence region). We want to restrict all relative output prices to belong to the set of ratios of all realized components in this random vector \( \mathbf{e} \) constrained to this set of realizations of high likelihood. This is achieved by imposing the requirement that the probability of the event \( \frac{e_i}{e_j} \leq \frac{w_i}{w_j} \) must belong to some narrow interval around 0.5 for each \( (i, j) \in IJ \). The theory of chance constrained programming provides a framework for this type of constraints. Two different approaches for modeling the desired probabilistic requirement are suggested with focus on the marginal distributions of \( e_i \) and \( e_j \) in approach #1 and on the joint distribution of \( e_i \) and \( e_j \) in approach #2. Approach #1 maintains the representation of the probabilistic constraints as simple as possible which offers computational advantages; the set of virtual multipliers is in this case constrained to a cone spanned by the Cartesian product of confidence intervals. The focus on the joint distribution of \( e_i \) and \( e_j \) in approach #2 offers the advantage that the assurance cone constraints are supporting hyperplanes to confidence regions for the joint distributions for all pair of prices, if

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5 The specification of ARs in the form of smooth cone constraints is subject to ongoing research the results of which are to be reported in a separate paper which may be seen as a Part II compared to the current one on polyhedral cone constraints.

Smooth cone constraints emerge when focus is on the \( s \)-dimensional joint distribution of the stochastic benchmark vector contrary to the focus on marginal distributions in this paper. Focusing on the \( s \)-dimensional joint distribution we do not to accept a dual price vector \( \mathbf{u} \) if the probability of observing and scaled vector \( \mathbf{e} \) different from the \( \mathbf{u} \) vector is small. However, since we compare an \( s \)-dimensional vector \( \mathbf{e} \) with a \( s \)-dimensional vector \( \mathbf{u} \) directly, we need a metric to measure the norm of these two vectors. Based on the chosen metric we constrain the norms of the two vectors to be close to each other.

6 Observe that the "norm-vector" \( \mathbf{e} \) degenerates into a price vector known with certainty, in the case where the variation in each component of \( \mathbf{e} \) is equal to zero. Hence, the standard approach for measurement of allocative efficiency can be considered a special case within the framework to be suggested. We consider for this reason the use of a stochastic "norm-vector" in the definition of probabilistic ARs highly consistent with standard DEA.
these regions are convex sets. Furthermore, explicit links exist between the probability levels for the confidence regions and the imposed chance constraints, if the distributional assumptions are derived from a multivariate normal distribution.

To simplify the presentation of the probabilistic AR’s we depart from the general case and in the rest of the paper restrict our attention to cases which satisfy the following assumptions: Assume that $\mathbf{e}$ is distributed according to a symmetric multivariate distribution, where all marginal distributions are identical and only parametrized by a location and a scale parameter. Let $\overline{\mathbf{e}} \equiv [\overline{e}_1, \ldots, \overline{e}_s]^\top$ be the vector of mean values of the vector $\mathbf{e}$, let $(\sigma_1^2, \ldots, \sigma_s^2)^\top$ be the vector of variances and assume that $\text{Cov}(e_i, e_j) = 0$, $i \neq j$.\footnote{These simplifying assumptions are only employed to avoid more complex expressions of the probabilistic ARs. In the hospital application mentioned above more flexible distributional assumptions have been employed.} We will now restrict all relative output prices by only allowing $u_i$ and $u_j$ to belong to the cone generated by the confidence intervals for $e_i$ and $e_j$, $(i, j) \in IJ$. Assuming that the relevant confidence intervals are $[\overline{e}_i + \eta_i^- \sigma_i, \overline{e}_i + \eta_i^+ \sigma_i]$ for some fractiles $\eta_i^-$, $\eta_i^+$, $i = 1, \ldots, s$ this can be accomplished by restricting $\mathbf{u}$ to belong to the following polyhedral cone $\overline{C}_2$:\footnote{In fact, we want to restrict $\mathbf{u}$ to the cone $\overline{C}_2 \cap \mathbb{R}^4_+$. In the rest of the paper we will implicitly assume that assurance cones are subsets of the $s$ - dimensional positive orthant.}

$$\mathbf{u} \in \overline{C}_2([\overline{e}_1, \ldots, \overline{e}_s]^\top, [\sigma_1^2, \ldots, \sigma_s^2]^\top, [\eta_1^-, \ldots, \eta_s^-]^\top, [\eta_1^+, \ldots, \eta_s^+]^\top) \equiv (2)$$

$$\left\{ \mathbf{u} \mid \frac{\overline{e}_i + \eta_i^+ \sigma_i}{\overline{e}_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\overline{e}_i + \eta_i^- \sigma_i}{\overline{e}_j + \eta_j^+ \sigma_j}, (i, j) \in IJ \right\}$$

Figure 1 illustrates this cone for the case of $s = 3$, $\overline{e}_i = 60$, $\sigma_i = 20$, $\eta_i^- = -1$, $\eta_i^+ = 1$. The gray squares in each of the three projections in figure 1B correspond to the Cartesian product of the confidence intervals $[40, 80]$ for $e_i$, $i = 1, 2, 3$. 

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Figure 1A. The cone \( \overline{C}_2() \) in three dimensions. \( \bar{z}_i = 60, \sigma_i = 20, \eta_i^- = -1, \eta_i^+ = 1, i = 1, 2, 3 \)

Figure 1B. The cone \( \overline{C}_2() \) in the three projections. The gray squares in each of the three projections in figure 1B correspond to the Cartesian product of the confidence intervals [40,80]

Figure 1. Assurance Regions in Multiplier Space

Hence, the cone in Figure 1A is \( \overline{C}_2([60,60,60]^T, [20,20,20]^T, [-1, -1, -1]^T, [1,1,1]^T) \). It is generated as the intersection of six halfspaces and has six extreme rays:

\[
\overline{C}_2([60,60,60]^T, [20,20,20]^T, [-1, -1, -1]^T, [1,1,1]^T) \equiv
\]

\[
\{(u_1, u_2, u_3)^T \mid C(u_1, u_2, u_3)^T \geq 0\}
\]

\[
C = \begin{bmatrix}
-1 & 2 & 0 \\
2 & -1 & 0 \\
-1 & 0 & 2 \\
2 & 0 & -1 \\
0 & -1 & 2 \\
0 & 2 & 1
\end{bmatrix}
\]

We have the three cones \( \text{cone}(a_1, a_2) \), \( \text{cone}(b_1, b_2) \) and \( \text{cone}(c_1, c_2) \) centered around \((e_1, e_2, 0), (e_1, 0, e_3)\) and \((0, e_2, e_3)\) in the three projections in Figure 1B.

Observe that (2) specifies ARs based on a probabilistic statement derived from the distributional characteristics of the stochastic norm vector \( \epsilon \). In fact, (2) can be expressed as the certainty equivalent to an assurance region based on a system of chance constraints. Land, Lovell and Thore (1993) and Olesen and Petersen (1995) propose two ways of incorporating stochastic inputs and outputs into a non-parametric (DEA-like) efficiency analysis. We will now proceed along the same lines to use a stochastic norm vector to define chance constrained assurance cones in multiplier space.

Let us return to the assurance region \( \overline{C}_2() \) defined by (2). Formulation (2) was suggested as a simple and intuitively appealing approach for imposing the requirement that the ratios between each pair of multipliers must belong to a set of all corresponding ratios from realizations of the stochastic benchmark vector considered of high likelihood. It is now to be shown that (2) can be given a theoretical foundation within the theory of chance constrained programming. This result is established by demonstrating that \( \overline{C}_2() \) is a so-called certainty equivalent for a set of probabilistic constraints defined in relation to the stochastic benchmark vector \((e_1, \ldots, e_s)\). Consider for each \((i,j) \in IJ\) the probabilistic statement:

\[
\text{The probability of each of the two events } \frac{c_{ij}}{u_i} \leq c_{ij} \text{ and } c_{ij} \leq \frac{c_{ij}}{u_j} \text{ must belong to the}
\]
The certainty equivalents for the set of probabilistic constraints in (3) are

\[
\frac{c_{ij}u_i - \bar{\xi}_i}{\sigma_i} \in [\eta_i^-, \eta_i^+], \quad \frac{c_{ij}u_j - \bar{\xi}_j}{\sigma_j} \in [\eta_j^-, \eta_j^+]
\]

(4)

for some \( c_{ij} \in \mathbb{R}_+ \), \( \forall (i, j) \in IJ \)

where \( \eta_k^- \equiv \Phi^{-1}(\alpha_k^-), \eta_k^+ \equiv \Phi^{-1}(\alpha_k^+) \), \( k = i, j \), and \( \Phi(.) \) is the distribution function for the distribution with zero mean and variances equal to one. It is easily seen that (5) is implied by (4):

\[
\frac{\bar{\xi}_i + \eta_i^+ \sigma_i}{\bar{\xi}_j - \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\bar{\xi}_i + \eta_i^- \sigma_i}{\bar{\xi}_j - \eta_j^+ \sigma_j}, \quad \forall (i, j) \in IJ,
\]

(5)

Let \( u \) denote any vector of multipliers satisfying (5). For each pair of elements \((u_i, u_j)\) in \( u \), \((i, j) \in IJ \), define \( c_{ij} \) such that

\[
\max\left(\frac{\sigma_i + \eta_i^- \sigma_i}{u_i}, \frac{\sigma_j - \eta_j^- \sigma_j}{u_j}\right) \leq c_{ij} \leq \min\left(\frac{\sigma_i + \eta_i^+ \sigma_i}{u_i}, \frac{\sigma_j - \eta_j^+ \sigma_j}{u_j}\right), \quad \forall (i, j) \in IJ
\]

It is easily seen that any \( u \) satisfying (5) also satisfies (4) with this choice of \( c_{ij} \). Hence (4) and (5) are equivalent representations of the same cone.
The following two cases may occur in the definition of the set of cone constraints by (5):

\( i) \quad 0.5 - \alpha_i^- = \alpha_i^+ - 0.5 \) for all \( i \)

\( ii) \quad 0.5 - \alpha_i^- \neq \alpha_i^+ - 0.5 \) for some \( i \)

\( i) \) is the standard case with two sided confidence intervals determined with equal probability mass on each side of the mean values. It is demonstrated below that (5) in this case defines a cone spanned by the Cartesian products of the confidence intervals. \( ii) \) is the case of confidence intervals, where probability masses of unequal size are cut off in the ends of a 2-sided confidence interval. The cone defined by (5) is in this case no longer spanned by the Cartesian product of the partial confidence intervals and it is affected by the order in which the \( s \) prices are represented in the set \( IJ \), see Appendix.

Focus is in this section on case \( i) \) because this is the case related to the standard definition of confidence intervals. The assumption \( 0.5 - \alpha_i^- = \alpha_i^+ - 0.5 \) for all \( i \) maintained in case \( i) \) implies along with the assumption that the distributions of the components of \( e \) are symmetric that (4) can be rewritten as

\[
\frac{\varepsilon_i + \eta_i^+ \sigma_i}{\varepsilon_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\varepsilon_i + \eta_i^- \sigma_i}{\varepsilon_j + \eta_j^+ \sigma_j}, \forall (i, j) \in IJ,
\]

(6) is identical to formulation (2) suggested as a simple and intuitively appealing approach for imposing the requirement that the ratios between each pair of multipliers must belong to a set of all corresponding ratios from realizations of the stochastic benchmark vector considered of high likelihood. Hence, letting \( \eta_i^- = \Phi^{-1}(\alpha_i), \eta_i^+ = -\Phi^{-1}(\alpha_i), \) for \( \alpha_i \in (0,0.5), i = 1, \ldots, s \), \( \tilde{C}_2(\cdot) \) in (2) is a certainty equivalent for the cone \( C_2(\cdot) \):

\[
C_2([\alpha_1, \ldots, \alpha_s]^T) \equiv \begin{cases} 
\{ u \mid 1 - \alpha_i \geq Prob\left(\frac{\varepsilon_i}{u_i} \leq c_{ij}\right) \geq \alpha_i, 1 - \alpha_j \geq Prob\left(\frac{\varepsilon_j}{u_j} \geq c_{ij}\right) \geq \alpha_j, \right. 
\text{for some } c_{ij} \in \mathbb{R}_+, (i, j) \in IJ \}
\end{cases}
\]

(7)

The equivalence between cones \( \tilde{C}_2(\cdot) \) and \( C_2(\cdot) \) is established by assigning each pair of multipliers \( (u_i, u_j), (i, j) \in IJ \), a scaling factor \( c_{ij} \) of its own. Otherwise, we would have (5) implied by (4) but not vice versa. The scaling factors \( c_{ij} \) are not necessarily identical for different pairs of multipliers. However, it is possible to find at least one common \( c \in \mathbb{R}_+ \) such that \( c_{ij} = c, (i, j) \in IJ, \) if \( 0.5 - \alpha_i^- = \alpha_i^+ - 0.5 \) for all \( i \) as maintained in this section. The existence of a common scaling factor \( c \) can be demonstrated as follows:
Theorem 1: Let \( A \) and \( B \) be

\[
A \equiv \left\{ u \in \mathbb{R}^n_+ : \frac{\bar{e}_i + \eta^+_i \sigma_i}{\bar{e}_j + \eta^+_j \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\bar{e}_i + \eta^-_i \sigma_i}{\bar{e}_j + \eta^-_j \sigma_j}, \forall (i, j) \in IJ \right\}
\]

\[
B \equiv \left\{ u \in \mathbb{R}^n_+ : \frac{c u_i - \bar{e}_i}{\sigma_i} \in [\eta^-_i, \eta^+_i], \frac{c u_j - \bar{e}_j}{\sigma_j} \in [\eta^-_j, \eta^+_j] \right\}
\]

\[
\forall (i, j) \in IJ, \ for \ some \ c \in \mathbb{R}_+
\]

Then \( A = B \).

Proof: See Appendix.

The existence of a common scaling factor \( c \) implies that the constraints imposed on the vector of virtual multipliers \( \mathbf{u} \) by (2) can be given the interpretation of a requirement that \( \mathbf{u} \) must belong to a set of realizations of the stochastic benchmark vector \( \mathbf{e} \) so that \( c \mathbf{e}, c \in \mathbb{R}_+ \), is considered of high likelihood.

Theorem 1 implies that the set of multipliers satisfying the cone constraint (2) can be given the interpretation as a simple scaling of multipliers in the set defined by the Cartesian product of confidence intervals. Observe that a distinction between \( \eta^-_i \) and \( \eta^+_i \) is maintained in the proof, although we have assumed that \( \eta^-_i = -\eta^+_i \) for all \( i \). The distinction is maintained in order to show that the interpretation remains valid for (2) but is not valid for (5) in the case of asymmetric confidence intervals. In fact, the case i) assumption \( \eta^-_i = -\eta^+_i \) for all \( i \) is from a formal point of view maintained because we want (5) to be the certainty equivalent for (3).

Let \( t_i \) and \( u_i, i = 1, ..., s \), denote the lower and upper confidence limit, respectively. By Theorem 1, the cone \( \overline{C}_2(\cdot) \) can be written

\[
\overline{C}_2(\cdot) = \{ \mathbf{u} : c u_i = \mu_i t_i + (1 - \mu_i) u_i, \mu_i \in [0, 1], i = 1, ..., s \}
\]

with \( c \in \mathbb{R}_+ \). This expression defines the cone by a scaling of points in the confidence region defined by the Cartesian product of all confidence intervals. The formulation is in practical applications preferable compared to (2) since it only involves \( 2 s \) linear equations as opposed to the \((s - 1) \times (s - 2)\) inequalities needed in (2).10

The cone \( C_2 \) and its equivalent representation \( \overline{C}_2(\cdot) \) was obtained by imposing the intuitively appealing requirement that the set of virtual multipliers must be chosen so that all relative prices

\[10 \text{ It should be observed that the results reported above can be derived by defining confidence intervals directly in terms of their lower and upper limits. The chance constraint programming framework does not require confidence intervals to be defined by deviations compared to a mean value as in the text above. This way of writing confidence intervals has only been adopted in order to ease exposition and maintain a close relationship to the standard exposition in chance constraint programming.} \]
belong to the cone generated by the Cartesian product of confidence intervals at probability levels $\alpha$ for the corresponding components $e_i$ and $e_j$ in the stochastic benchmark vector for all $(i, j) \in IJ$. Hence, focus is on the marginal distributions of $e_i$ and $e_j$ and not the joint distributions of the two random variables. Theil (1971 p. 132) shows that the rectangle formed by this Cartesian product provides a confidence region with probability at least $1 - 2\alpha$. Clearly, a more "correct" approach is to construct this confidence region directly from the joint distribution of $e_i$ and $e_j$, as the region with the smallest "volume" that contains a specified proportion $\gamma_{ij}$ of the distribution. Assume that the joint distribution is absolute continuous and derived from a multivariate normal distribution. Denote the density function $f(e_i, e_j)$. This density function allows for the definition of a confidence region $D_{ij}(\gamma_{ij}), (i, j) \in IJ$, at probability level $\gamma_{ij}$ for joint realizations of $e_i$ and $e_j$. For this case a confidence region at level $\gamma_{ij}$ is by construction a set of realizations of $e_i$ and $e_j$ with the properties

i) realizations of $(e_i, e_j)$ inside the confidence region will be observed in $\gamma_{ij}\%$ of all instances, and

ii) $f(e_i^1, e_j^1) \geq f(e_i^2, e_j^2)$ for any two pairs $(e_i^1, e_j^1) \in D_{ij}(\gamma_{ij})$ and $(e_i^2, e_j^2) \notin D_{ij}(\gamma_{ij})$.

Hence, it may be argued that confidence regions define joint realizations of high likelihood for the random variables $e_i$ and $e_j$ and are more appropriate compared to the Cartesian products of confidence intervals, because the meaning of joint realizations of high likelihood are defined by properties i) and ii).

Consider the cone

$$\overline{C}_3 = \{ u \ | \exists (e_i, e_j) \in D_{ij}(\gamma_{ij}) : (u_i, u_j) = c_{ij}(e_i, e_j) \text{ for some } c_{ij} \in \mathbb{R}_+, (i, j) \in IJ \}$$

$\overline{C}_3$ can be given the interpretation as an AR, since it can be rewritten as (1). The rewriting is not trivial since the supporting hyperplanes for the confidence regions involved passing through origin must be determined explicitly in order to identify the upper and lower bounds for the price ratios involved in (1). The cone $\overline{C}_3$ can also for some distributions be given a foundation within the theory of chance constrained programming by demonstrating its identity with a certainty equivalent for a set of probabilistic constraints defined in relation to the stochastic benchmark vector $(e_1, \ldots, e_s)$. Consider for each $(i, j) \in IJ$ the probabilistic statement:

The joint probability of the two events $\frac{\alpha_i}{u_i} \leq c_{ij} \leq \frac{\alpha_j}{u_j}$ must belong to the interval $[\alpha_{ij}^-, \alpha_{ij}^+]$ centered around 0.5 for some $c_{ij} \in \mathbb{R}_+$.

Johnson and Kotz (1972, p. 66-70) denote this set a tolerance region.
Thus, focus is at this point on the joint distribution of $e_i$ and $e_j$. The following probabilistic polyhedral cone assurance $C_3()$ is obtained:

$$C_3\left( [\bar{e}_i, \cdots, \bar{e}_s]^T, [\sigma_i, \cdots, \sigma_s]^T, [\alpha_{i2}, \cdots, \alpha_{i(s-1)}]^T, [\alpha^+_i, \cdots, \alpha^+_{i(s-1)}]^T \right) \equiv (8)$$

$$\{ u | \alpha^+_{ij} \geq Prob\left( \frac{e_i}{e_j} \leq \frac{u_i}{u_j} \right) \geq \alpha^-_{ij}, (i, j) \in IJ \} =$$

$$\{ u | Prob(e_iu_j - e_ju_i \leq 0) \geq \alpha^-_{ij}, Prob(e_ju_i - e_iu_j \leq 0) \geq 1 - \alpha^+_{ij}, (i, j) \in IJ \}$$

To simplify, we use the same probability level $p = \alpha^-_{ij} = (1 - \alpha^+_{ij})$ in all $s \times (s-1)$ chance constraints that generate this assurance cone $C_3()$. The cone $\overline{C}_3$ is the certainty equivalent for the assurance cone $C_3()$ given the maintained hypothesis that the components of the norm vector are i.i.d. jointly normal:

$$\overline{C}_3([\bar{e}_i, \cdots, \bar{e}_s]^T, [\sigma_i, \cdots, \sigma_s]^T, \eta^-, \eta^+ \equiv \Phi^{-1}(p), \eta^+ = \Phi^{-1}(1 - p), \Phi \text{ the standard normal distribution function. Observe that the constraints involved in the specification are non-linear. However, a linearization is available since the underlying chance constraints only involves two variables, see Kall (1976) for details. It follows by the discussion in Olesen and Petersen (1995) that the deterministic equivalent for a chance constraint } Prob\left( \frac{\bar{e}_i}{e_j} \leq \frac{u_i}{u_j} \right) \geq \pi, (i, j) \in IJ, \text{ is a supporting hyperplane for the confidence region } D_{ij}(\gamma) \text{ at probability level } \gamma \text{ for } \pi \geq 0.5 \text{ with the link between } \gamma \text{ and } \pi \text{ given by } \pi = \Phi(\epsilon) \text{ where } P\left( \chi^2_{(2)} \leq c^2 \right) = \gamma. \text{ Clearly, the probability level } p = \alpha^-_{ij} \text{ in (8) is below 0.5, since } \eta^- \leq 0. \text{ The chance constraint with probability level } p < 0.5 \text{ generates a supporting hyperplane to a confidence region in } \mathbb{R}^2 \text{ at probability level } \gamma \text{ identical to the one generated at probability level } 1 - p, \text{ i.e. } p = \Phi(\epsilon - c) \text{ with } P\left( \chi^2_{(2)} \leq (\epsilon - c)^2 \right) = \gamma. \text{ Thus, the certainty equivalent for the assurance cone } C_3() \text{ denoted } \overline{C}_3 \text{ is identical to } \overline{C}_3 \text{ thus providing a foundation for } \overline{C}_3 \text{ within the theory of chance constrained programming.}$$

Consider for a given $(i, j) \in IJ$ the pair of chance constraints that generate $\overline{C}_3()$:

$$[\bar{e}_i, \bar{e}_j][u_j, -u_i]^T \leq -\eta^- \sqrt{u_i^2 \sigma_j^2 + u_j^2 \sigma_i^2} \leq$$

$$\{ u | \frac{\bar{e}_i}{e_j} \leq \frac{u_i}{u_j} \} \text{ and to the event }$$

$$\left( \frac{u_i}{u_j} \leq e_{ij}, c_{ij} \leq \frac{u_i}{u_j} \right) \text{ is as follows: } P'(u_je_i - u_ie_j \leq 0) = \int_0^c F_{c, \sigma}(e_i \leq u_i; c_{ij}, e_j \geq u_j; c_{ij}) \, dc_{ij}$$
\[
[\hat{e}_i, \bar{e}_j] - u_j, u_i] \leq -\eta^+ \sqrt{u_i^2 \sigma_i^2 + u_j^2 \sigma_j^2}
\]

(10)

(11) is obtained by the shift of variables \(\hat{u}_i \equiv u_i \sigma_j \) and \(\hat{u}_j \equiv u_j \sigma_i\), \((i, j) \in IJ\):

\[
\frac{\left[ \sigma_i \sigma_j \right] [\hat{u}_j, -\hat{u}_i]^T}{\sqrt{\hat{u}_i^2 + \hat{u}_j^2}} \leq -\eta^- \quad \text{and} \quad \frac{\left[ \sigma_i \sigma_j \right] [-\hat{u}_j, \hat{u}_i]^T}{\sqrt{\hat{u}_i^2 + \hat{u}_j^2}} \leq -\eta^+
\]

Thus, (9) and (10) can be given an interpretation in terms of the following requirement on the maximum angle allowed between the mean values of the stochastic variables \(e_i\) and \(e_j\) and feasible \(u_i\) and \(u_j\):

\[
\text{angle} \left( \begin{bmatrix} \bar{e}_i \\ \bar{e}_j \end{bmatrix}, \begin{bmatrix} \hat{u}_i \\ \hat{u}_j \end{bmatrix}^T \right)
\]

\[
\begin{align*}
\leq & \quad \frac{\pi}{2} - \arccos \left( \frac{-\eta^-}{\sqrt{\left( \sigma_i \right)^2 + \left( \sigma_j \right)^2}} \right) \quad \text{if} \quad \frac{-\eta^-}{\sqrt{\left( \sigma_i \right)^2 + \left( \sigma_j \right)^2}} \in [-1, 1] \\
\in & \quad [0, 2\pi] \quad \text{otherwise}
\end{align*}
\]

(12)

Figure 2 illustrates this cone for \(s = 2\), \(\bar{e}_k = 60\), \(\sigma_k = 20\), \(\eta^- = -1\). The cone can be defined by considering the maximum angle between \(\bar{e}_k\) and \(\hat{u}_k\):

\[
\text{angle}([3, 3]^T, [\hat{u}_i, \hat{u}_j]^T) \leq 90^\circ - \arccos \left( \frac{1}{\sqrt{13}} \right) \approx 13.64^\circ
\]

allowed by (11) (or (12)). Alternatively, the cone can be seen as spanned by the confidence region \(D_{ij}(\gamma)\) generated by the probability level \(p = \Phi(1)\). Hence, \(p = 0.16\) and the corresponding \(\gamma\) is defined by \(\gamma = P\left(\chi^2_{(2)} \leq 1\right) \approx 0.4\), where \(\chi^2_{(n)}\) is a chi-square distributed stochastic variable with \(n\) degrees of freedom.\(^{13}\) \(D_{ij}(0.4)\) is shown as the gray disc spanning the cone.

where \(P'()\) is the probability measure for the random variable \(u_{j}e_i - u_{i}e_j\) and \(P_{e_i,e_j}()\) is the joint probability measure for the random vector \((e_i, e_j)\).

\(^{13}\) Observe that the confidence region may include the origin and extend into the negative orthant. To illustrate by example, we have a confidence region with the origin as a point on the boundary if \(\bar{e}_i = \sigma_i\) and \(\bar{e}_j = \sigma_j\) and \(-\eta^- = \sqrt{2}\) and the angle in (11) becomes \(\text{angle} \left( [1, 1]^T, [\hat{u}_i, \hat{u}_j]^T \right) \leq \frac{\pi}{2}\). If \(\bar{e}_i < \sigma_i\) and \(\bar{e}_j < \sigma_j\) and \(-\eta^- = \sqrt{2}\) then we have a confidence region with the origin as an interior point. Hence, we get no restriction on the vector \([\hat{u}_i, \hat{u}_j]^T\).
3. Polyhedral assurance cones generated from a priori information in the form of two stochastic target vectors.

The specification of lower and upper bound on the relative multipliers as two random one-sided "target-vectors" is the issue to be addressed in this section. It will be shown that the lower and upper bound parameters \( a_{i,j} \) and \( b_{i,j} \), \((i, j) \in IJ\), in (1) can be defined as a function of the left and right end-point of one-sided confidence intervals for the components in the two stochastic "target-vectors".

Alternative approaches for modeling the lower and upper bounds in an AR as random variables are available. A direct modeling of the lower and upper bounds in an AR as random variables may seem the most obvious procedure. Assume that \( a_{i,j} \) and \( b_{i,j} \), \((i, j) \in IJ\), are random with mean values \( \bar{a}_{i,j} \) and \( \bar{b}_{i,j} \) and variances \( \sigma^2_{a_{i,j}} \) and \( \sigma^2_{b_{i,j}} \), \((i, j) \in IJ\). The corresponding one-sided confidence intervals are \([\bar{a}_{i,j} + \eta^l_{i,j}\sigma_{a_{i,j}}, \infty]\) and \((-\infty, \bar{b}_{i,j} + \eta^u_{i,j}\sigma_{b_{i,j}}]\) for some fractiles \( \eta^l_{i,j} \) and \( \eta^u_{i,j} \) of opposite sign with \( l \) and \( u \) indicating lower and upper bounds, respectively. The following polyhedral assurance cone \( C_4() \) is easily obtained:

\[
C_4 = \left( [\bar{a}_{1,2}, \ldots, \bar{a}_{s-1,s}]^\top, [\bar{b}_{1,2}, \ldots, \bar{b}_{s-1,s}]^\top, [\sigma_{a_{1,2}}, \ldots, \sigma_{a_{s-1,s}}]^\top, \right.
\]
\[
\left[ \sigma_{h_1,2}, \ldots, \sigma_{h_{s-1},s} \right]^{T}, \left[ \eta_{1,2}^{w}, \ldots, \eta_{s-1,s}^{w} \right]^{T}, \left[ \eta_{1,2}^{w}, \ldots, \eta_{s-1,s}^{w} \right]^{T}
\]

\[\equiv \left\{ \mathbf{u} \mid \overline{a}_{i,j} + \eta_{i,j}^{w} \sigma_{a_{i,j}} \leq \frac{u}{w_{j}} \leq \overline{b}_{i,j} + \eta_{i,j}^{w} \sigma_{b_{i,j}}, (i,j) \in IJ \right\}\]

Clearly, \(C_4(\cdot)\) is a certainty equivalent for the following probabilistic cone \(C_4(\cdot)\): \(^{14}\)

\[C_4 = \left( \left[ \overline{a}_{1,2}, \ldots, \overline{a}_{s-1,s} \right]^{T}, \left[ \overline{\eta}_{1,2}^{w}, \ldots, \overline{\eta}_{s-1,s}^{w} \right]^{T}, \left[ \sigma_{a_{1,2}}, \ldots, \sigma_{a_{s-1,s}} \right]^{T}, \left[ \eta_{1,2}^{w}, \ldots, \eta_{s-1,s}^{w} \right]^{T} \right)
\]

\[\equiv \left\{ \mathbf{u} \mid \text{Prob}(a_{i,j} \leq \frac{u}{w_{j}}) \geq \alpha(\eta_{i,j}^{w}), \text{Prob}(\frac{u}{w_{j}} \leq b_{i,j}) \geq \alpha(\eta_{i,j}^{w}), (i,j) \in IJ \right\}\]

The approach is characterized by imposing stochastic bounds on the ratios between multipliers directly and is for this reason appropriate in instances with a priori information concerning maximum and minimum rates of substitution and their variation.\(^{15}\)

Assume that a priori information relates to a random vector of lower and upper bounds. Clearly, lower and upper bounds are defined uniquely to within scalar multiplication as is the variation in prices. It is in this case more appropriate to model the stochastic variation in the components defining the bounds directly as follows. Assume that we have two random \(s\)-dimensional lower bound and upper bound “target-vectors” \(l\) and \(u\). Assume that both \(l\) and \(u\) are distributed according to two symmetric multivariate distributions, where all marginal distributions are identical and only parametrized by a location and a scale parameter. Let \(\overline{l}\) and \(\overline{u}\) be the mean vectors of the two vectors, let \((\sigma_{a_{1,1}}, \ldots, \sigma_{a_{s,s}})^{T}\) and \((\sigma_{a_{1,1}}, \ldots, \sigma_{a_{s,s}})^{T}\) be the two vectors of variances and assume that \(\text{Cov}(l_{i},w_{j}) = 0, \forall i, j, \text{Cov}(l_{i},l_{j}) = \text{Cov}(w_{i},w_{j}) = 0, i \neq j\). As in section 2 we can either constrain the multipliers to a cone spanned by the Cartesian product of the two confidence intervals for each pair of prices or we can constrain the multipliers to a cone spanned by the two dimensional confidence region for each pair of prices. Observe that confidence intervals at this point are one-sided, since \(l\) \((u)\) imposes no upper \((lower)\) bounds on prices.

Consider the restriction of all relative output prices by only allowing \(u_{i}\) and \(u_{j}\) in the cone generated by one-sided confidence intervals for \(l_{i}\) and \(l_{j}\) and for \(w_{i}\) and \(w_{j}\), \((i,j) \in IJ\). The relevant one-sided confidence intervals can be written \(\left[ \overline{l}_{i} + \eta_{i}^{l} \sigma_{l_{i}}, -\infty \right)\) and \(\left( -\infty, \overline{w}_{i} + \eta_{i}^{w} \sigma_{w_{i}} \right]\) for some fractiles \(\eta_{i}^{l}\) and \(\eta_{i}^{w}\) of opposite sign, \(i = 1, \ldots, s\), where the sign of \(\eta_{i}^{l}\) and \(\eta_{i}^{w}\) is different, for all \(i\). Observe that fractiles are not labeled with superscripts \(+/-\) as in Section 2, since confidence intervals are one-sided. The desired restriction is imposed by the requirement that \(\mathbf{u}\) must belong to the following polyhedral cone \(\overline{C}_{5}(\cdot)\)

\(^{14}\) Observe that probability levels \(\alpha(\cdot)\) are dependent on the chosen fractiles. It is easily verified that \(\alpha(\eta_{i,j}^{w}) = \Phi(\eta_{i,j}^{w}), k = l, w\), in the very simple case under consideration.

\(^{15}\) Observe that the benchmark approach suggested in Section 2 also is available in the case of a priori information on rates of substitution.
Consider for each \((i,j)\in IJ\) the probabilistic statement:

i) The probability of each of the two events \(\frac{l_i + \eta_i^k \sigma_k}{u_i} \leq c_{ij} = \frac{w_i + \eta_i^w \sigma_w}{u_j}\) must not fall below \(\alpha_i^k\) and \(\alpha_j^w\), respectively, and

ii) The probability of each of the two events \(\frac{l_i + \eta_i^k \sigma_k}{u_i} \leq c_{ij} = \frac{w_i + \eta_i^w \sigma_w}{u_j}\) must not fall below \(\alpha_i^k\) and \(\alpha_j^w\), respectively, for some \(c_{ij} \in \mathbb{R}_+\).

The set of virtual multipliers \(u\) satisfying this statement can be written as the probabilistic cone \(C_5()\):

\[
C_5\left([\eta_1^l, \ldots, \eta_s^l]^T, [\eta_1^w, \ldots, \eta_s^w]^T\right) = \left\{ u \mid \text{Prob}\left(\frac{l_i}{u} \leq c_{ij} \leq \frac{w_i + \eta_i^w \sigma_w}{u_j}\right) \geq \alpha_i^k(\eta_i^k) \text{ and } \alpha_j^w(\eta_j^w) \right\}
\]

\(\overline{C}_5()\) is a certainty equivalent for \(C_5()\) with probabilities \(\alpha_i^k(\eta_i^k), i = 1, \ldots, s, k = l, w,\) chosen at appropriate levels thus establishing a statistical foundation for (13). Clearly, we must have \(\overline{l_i} + \eta_i^k \sigma_k \leq w_i + \eta_i^w \sigma_w\), \(i = 1, \ldots, s,\) in order to obtain a non-empty set of feasible multipliers.

Figure 3 illustrates the assurance cone \(\overline{C}_5()\) for \(s = 2, \ l = (40, 40), \ w = (80, 80), \ \sigma_k = \sigma_w = 20\) and \(\eta_i^k = -\eta_i^w = 0.5, k = i, j\). The areas marked by the gray areas indicate the Cartesian product of the confidence intervals for \((l_i\) and \(w_j)\) and for \((\eta_i^l\) and \(\eta_j^w)\), and the lower (upper) bound for the ratio \(\frac{w_i}{u_j}\) is defined by the slope of the lower (upper) extreme ray.

Observe that the probability levels to be used in the two pairs of chance constraints if the above cone is to be generated are dependent on the imposed fractile levels. It is easily seen that \(\alpha_i^k(\eta_i^k), i = 1, \ldots, s, k = l, w,\) must be below 50% with \(\eta_i^k = -\eta_i^w = 0.5, k = i, j.\)
Figure 3. Assurance regions $\mathcal{C}_5(\cdot)$ in the multiplier space generated from two sets of Cartesian products of confidence intervals. The assurance cone $\mathcal{C}_5(\cdot)$ is illustrated for $s = 2$, $\bar{I} = (40,40)$, $\bar{w} = (80,80)$, $\sigma_{l_k} = \sigma_{w_k} = 20$ and $\eta^l_k = \eta^w_k = 0.5$, $k = i, j$. The areas marked by gray $[\bar{l}_i + \eta^l_j \sigma_{l_i}, \infty) \times (-\infty, \bar{w}_j + \eta^w_j \sigma_{w_j}]$ and $(-\infty, \bar{w}_i + \eta^w_i \sigma_{w_i}] \times [\bar{l}_j + \eta^l_j \sigma_{l_j}, \infty)$ indicate the two Cartesian products of the confidence intervals.

The spanning of the assurance cone of virtual multipliers based upon the two-dimensional confidence regions for $(l_i, w_j)$ and $(w_i, l_j)$, $(i, j) \in IJ$, has the advantage that the meaning of joint realizations of high likelihood are defined by properties i) and ii), see Section 2. Focus is for this reason next on constraining the multipliers to a cone spanned by confidence regions for $(l_i, w_j)$ and $(w_i, l_j)$, $(i, j) \in IJ$. The probabilities of the events $\frac{L_i}{W_j} \leq \frac{w_i}{w_j}$ and $\frac{w_i}{w_j} \leq \frac{w_i}{L_j}$ must in this case not fall below some level to be specified a priori. The following probabilistic polyhedral cone $C_6(\cdot)$ is obtained:

$$C_6([\alpha^1_{i_1} \ldots \alpha^k_{i_{(s-1)/2}}]^t k \in \{w, l\}) \equiv$$

$$\left\{ u \mid \text{Prob} \left( \frac{l_i}{w_j} \leq \frac{u_i}{u_j} \right) \geq \alpha^l_{ij}, \text{ and } \text{Prob} \left( \frac{w_i}{l_j} \geq \frac{u_i}{u_j} \right) \geq \alpha^w_{ij}, (i, j) \in IJ \right\} =$$

$$\left\{ u \mid \text{Prob}(l_iu_j - w_ju_i \leq 0) \geq \alpha^l_{ij}, \text{ Prob}(l_ju_i - w_iu_j \leq 0) \geq \alpha^w_{ij}, (i, j) \in IJ \right\}$$

Assume for reasons of exposition $p = \alpha^l_{ij} = \alpha^w_{ij}$ in all $s \times (s-1)$ chance constraints generating the assurance cone $C_6(\cdot)$ and that the components of the target vectors are i.i.d. joint
normal as above. The assurance cone \( C_0() \) can then be transformed to its certainty equivalent \( \overline{C}_0() \):

\[
\overline{C}_0([\bar{l}_i, \ldots, \bar{l}_s]^\top, [\bar{w}_i, \ldots, \bar{w}_s]^\top, [\sigma_{li}, \ldots, \sigma_{ls}]^\top, [\sigma_{wi}, \ldots, \sigma_{ws}]^\top, \eta) = \left\{ \begin{array}{l}
\bar{l}_i u_j - \bar{w}_j u_i + \eta \sqrt{u_i^2 \sigma_{ij}^2 + u_j^2 \sigma_{ji}^2} \leq 0, i = 1, \ldots, s, j = i + 1, \ldots, s \\
\{ u | \bar{l}_j u_i - \bar{w}_i u_j + \eta \sqrt{u_i^2 \sigma_{ij}^2 + u_j^2 \sigma_{ji}^2} \leq 0, i = 1, \ldots, s, j = i + 1, \ldots, s \}
\end{array} \right\} \cap
\]

where \( \eta = \Phi^{-1}(p) \), \( \Phi \) the standard normal distribution function.\(^{16}\)

Figure 4 illustrates this cone for \( s = 2 \), \( \bar{l} = (40, 40), \bar{w} = (80, 80), \sigma_{lk} = \sigma_{wk} = 20, k = i, j, \eta = 0.5 \). The gray ellipsoids indicate the confidence regions for \( (l_i, w_j) \) and \( (w_i, l_j) \) at probability level 0.12, since \( P(\chi^2_{(2)} \leq 0.25) \approx 0.12 \), where \( \chi^2_{(n)} \) is a chi-square distributed stochastic variable with \( n \) degrees of freedom. Hence, the probability \( \eta = \alpha_{ij} = \alpha_{ji}^+ \) in the chance constraints is equal to \( 1 - \Phi(0.5) \approx 1 - 0.69 = 0.31 \), i.e. about 31% of the probability mass for the distributions of \( (l_i, w_j) \) \([w_i, l_j]\) is located above the upper [below the lower] ray of the cone.

Figure 4. Assurance regions \( \overline{C}_0() \) in multiplier space generated from two dimensional joint confidence regions. The cone \( \overline{C}_0() \) is illustrated for \( s = 2 \), \( \bar{l} = (40, 40), \bar{w} = (80, 80), \sigma_k = 20, \eta = 0.5 \). The gray ellipsoids indicate the confidence regions.

4. Pairwise constraints versus constraints on all virtual multipliers.\(^{17}\)

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\(^{16}\) A linear specification is available as discussed above.

\(^{17}\) A more elaborated version of Chapter 4 is available from:

http://www.busieco.ou.dk/~ole/homepage.htm
Assurance regions defined as pairwise constraints on the virtual multipliers were introduced by Thompson et al. (1990) and Charnes et al. (1990) and further analyzed by Charnes et al. (1991). Analysts often refer to the notion of an expert opinion or to historical data when specifying the relevant range for each pair of relative prices. Unfortunately, an expert opinion is in many applications not available or too many different opinions make it difficult to use the information for construction of ARs. The analyst must in this case either rely on historical data and the inherent problems caused by this approach or choose some reasonable specification of the ranges and then vary the parameters to investigate the sensitivity of the results with regard to the specification of ranges.

It is often difficult to obtain an expert's opinion on the simultaneous variation in more than a few different prices and usually considered easier to obtain estimates on a reasonable range for each price ratio. Hence, posing questions on each of the many different relative prices is often a constructive way to get an assurance region generated from expert opinions. Unfortunately, the construction of assurance regions based on pairwise constraints results in a set of feasible multipliers subject to some inherent undesirable characteristics that are not immediately brought to the analyst's attention. Consider as an illustration the cone $\{a_{i,j}, b_{i,j}, (i, j) \in IJ\}$ with $a_{i,j} = \frac{1}{2}, b_{i,j} = 2 (i, j) \in \{(1, 2)(1, 3)(2, 3)\}$, which is identical to the one shown in Figure 1. An immediate but inaccurate intuition concerning this cone may indicate that it is symmetric around the vector $[1, 1, 1]$. Hence, we could hypothesize that constraining multipliers to this cone implies that we allow symmetric deviations from $[1, 1, 1]$ in all directions. However, the symmetry only holds true in each of the three two dimensional projections.

Let us consider the set of feasible $u_1, u_2$ and $u_3$, given $u_1 - u_2 = 0$. This situation is illustrated in Figure 5. The projection of the cone of feasible prices into the hyperplane characterized by $u_1 - u_2 = 0$ contains two of the six extreme rays spanning the cone of feasible multipliers, $(1,1,2)$ denoted $r_1$ and $(2,2,1)$ denoted $r_2$. The mean vector $(1,1,1)$ is also contained in this projection. Figure 5b illustrates that the mean vector is not located in the center of the cone. In fact, the angle $\alpha$ between the mean vector and $r_1$ is larger than the angle $\beta$ between the mean vector and $r_2$.\(^{18}\)

\(^{18}\) Obviously, the concept of angle between vectors is not units invariant but allows for an easy demonstration of the point to be made.
(u_1 + u_2) / \sqrt{2}

\alpha = \arccos\left(\frac{4}{\sqrt{3\sqrt{6}}}\right) \approx 0.34

\beta = \arccos\left(\frac{5}{3\sqrt{3}}\right) \approx 0.28

Figure 5A. Two extreme rays in the same projection.  
Figure 5B. The three vectors in the two dimensional projection

Figure 5: Feasible set of u_1, u_2 and u_3, given u_1 - u_2 = 0

The difference between these angles will approach zero with an increasing number of outputs, but the rate of convergence is slow if a_{i,j} is small and/or b_{i,j} is large in the cone constraints in (1), see Appendix.

Did the analyst have the situation in Figure 5 in mind when specifying the assurance region \( C_1(\frac{1}{2}[1,1,1], 2[1,1,1]) \)? Is it on purpose that the deviation of the price vector from the vector \([1,1,1]\) in the \(u_i,u_j\) projection is symmetric, \(i,j = (1,2),(1,3),(2,3)\) while the deviation in the \(u_i,(u_j+u_k)\) projection \(i,j,k = (1,2,3),(2,1,3),(3,1,2)\) is non-symmetric? We are of the opinion that an adjustment of the assurance region such that the two dimensional cones in all two dimensional projections containing the mean vector will be characterized by a symmetric deviation from the mean vector is of interest, if the answer to these questions is negative.

The geometric illustrations suggest that the extreme rays \(r_1, r_3\) and \(r_5\) should be pulled towards the mean vector so that the angles between the mean vector and the six extreme rays are made identical. However, the polyhedral cone of feasible multipliers can be expressed as the intersection of six halfspaces each of which bounded by a supporting hyperplane with one of the three components in the normal equal to zero. Three of these supporting hyperplanes contain the following three two-combinations of extreme rays \((r_1,r_2), (r_3,r_4)\) and \((r_5,r_6)\). The fact that one of the three components of the normal vector is equal to zero is a consequence of only including two prices in the inequalities determined from the bounds on the relative prices. Hence, if we pull the extreme rays \(r_1, r_3\) and \(r_5\) towards the mean vector then we will inevitably rotate the normal vectors from the three supporting hyperplanes such that no component is equal to zero. In other words, we cannot adjust the assurance region as desired.
unless more complex constraints are imposed on the virtual multipliers compared to those generated from bounding the relative prices as in (1).

A rotation of the unit vectors \( e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) \) around the mean vector \( (1,1,1) \) is one way to proceed if the cone of feasible virtual multipliers is to be characterized by a constant angle between the mean vector and the extreme rays. The rotation makes it possible to impose the same constraints as in \( C(I) = \{ (x, y, z) \in \mathbb{R}^3 : \frac{1}{2} [1, 1, 1], 2 [1, 1, 1] \} \) on prices expressed in a rotated basis. An explicit expression for the unit vectors rotated \( R(e_i, \gamma) \), \( i = 1, 2, 3 \), is provided by Theorem 3 in Appendix. It is also shown that the following assurance region is obtained if we impose the same constraints as in \( C(I) = \{ (x, y, z) \in \mathbb{R}^3 : \frac{1}{2} [1, 1, 1], 2 [1, 1, 1] \} \) on the prices expressed in all rotated bases:

\[
\tilde{C}_1(\frac{1}{2}, 2) \equiv \left\{ u \mid \frac{1}{2} \leq \frac{R(e_i, \gamma)}{R(e_j, \gamma)} u \leq 2, \ (i, j) = (1, 2), (1, 3), (2, 3), \ \gamma \in \mathbb{R}_+ \right\} \\
= \left\{ u \mid \text{angle}((1, 1, 1)^T, u) \leq \frac{\pi}{2} - \arccos\frac{1}{\sqrt{3} \sqrt{3}} \right\} \quad (17)
\]

Hence, a requirement of symmetric deviations from the mean vector leads to a non-polyhedral convex cone \( \tilde{C}_1(\frac{1}{2}, 2) \) generated from infinitely many extreme rays. This type of smooth cones and its potential use for specifying ARs is subject to ongoing research.

5. Concluding remarks.

Different approaches for introducing probabilistic assurance regions on the virtual multipliers in DEA have been analyzed and an inherent problem in the specification of ARs based upon pairwise bounds on the virtual multipliers is highlighted.

Section 2 can be seen as concerned with the case of uncertain price information. The AR approach allows for the incorporation of price information under conditions of no uncertainty by the set of constraints \( \frac{u_i}{u_j} = \frac{\bar{e}_i}{\bar{e}_j} \) for all \( (i, j) \in IJ \) where \( \bar{e}_i \) and \( \bar{e}_j \) denote prices of outputs \( i \) and \( j \). This approach is equivalent to the measurement of the allocative efficiency, since an aggregation of the \( s \) outputs into a single output by means of the price vector is implied. However, an estimation of the allocative efficiency may be of limited value in practical applications since prices may be subject to random variations, see Cooper et al. (1996). It may for this reason be necessary to model the random variation in prices in order to maintain the validity of the model. The approach suggested in Section 2 is one way to proceed. Prices are in this approach represented by a stochastic benchmark vector \( e = [e_1, ...., e_s]^T \) where each component \( e_i, \ i = 1, \ldots, s, \) is a random variable with the \( s \) random variables distributed according to some multivariate distribution. It is shown that the modeling of the requirement that the probability of the event \( \frac{u_i}{u_j} \leq \frac{u_i}{u_j} \) must belong to some narrow interval around 0.5 for each \( (i,j) \in IJ \) by chance constrained programming allows for the construction of assurance region cones. The AR-cones have a firm probabilistic foundation because each pair of
multipliers subjected to a scalar multiplication is required to belong to a set of realizations of the corresponding pair of stochastic prices considered of high likelihood.

We consider the situation discussed in Section 2 representative in a large number of applications because the case of uncertain price information frequently occurs. However, the suggested approach is not the only procedure available for incorporation of uncertain price information in DEA with assurance regions. The upper and lower bounds imposed on the relative prices may be considered the distinguishing feature of the standard AR-approach, and it is in some applications appropriate to consider the bounds themselves stochastic. This is the issue addressed in Section 3. The AR approach allows for the incorporation of lower and upper bounds on prices under conditions of no uncertainty by the set of constraints \( \frac{L_k}{w_j} \leq \frac{w_i}{w_j} \leq \frac{L_j}{w_i} \) for all \((i, j) \in IJ\) where \(L_k\) are lower and upper bounds on prices of outputs \(i\) and \(j\) known with certainty. However, the bounds may be subject to random variations which must be modeled in order to maintain the validity of the model. The approach suggested in Section 3 is one way to proceed. Lower and upper bounds are represented by two stochastic vectors \( l = [l_1, \ldots, l_s]^T \) and \( u = [u_1, \ldots, u_s]^T \) where each component \( l_i (u_i), i = 1, \ldots, s, \) is a random variable with the \(s\) random variables distributed according to some multivariate distribution. It is shown that the chance constrained programming modeling of the requirement that the probability of the event \( \frac{L_k}{w_j} \leq \frac{w_i}{w_j} \leq \frac{L_j}{w_i} \) must not fall below some level for each \((i, j) \in IJ\) allows for the construction of assurance region cones. The AR-cones have a firm probabilistic foundation because each pair of multipliers subjected to a scalar multiplication is required to belong to a set of realizations of the corresponding pair of stochastic prices considered of high likelihood.

Structure concerning the distributional characteristics for the stochastic variables has been imposed. It is important to be aware that the (simplifying) structural assumptions are maintained in order to ease exposition. It is possible to derive probabilistic assurance cones under conditions of more flexible distributional assumptions as witnessed by the hospital analysis by Olesen & Petersen (1998).

The construction of assurance regions based on pairwise constraints is subject to the inherent undesirable characteristic that it is not possible to identify a ray in the cone generated by a symmetric AR with the property that the angle between this ray and the set of extreme rays is the same. We are of the opinion that the existence of this characteristic is not obvious. It is shown in Section 4 that a three dimensional polyhedral assurance cone generated from symmetric pairwise constraints on the multipliers allow for a larger variation in some directions compared to others. The cone used in Section 4 can be generated by a symmetric specification of the flexible AR suggested by Thompson et al. (1997) or by symmetric deviations compared to a mean vector as in Section 2. The cone can therefore in certain two dimensional projections be expressed as generated by the mean values of the relevant components of the stochastic benchmark vector plus a vector of deviations with the lengths of the latter being less than or equal to some constant imposed as the only constraint. Thus, prices are allowed to deviate from a center vector in an arbitrary direction with the norm of the deviation vector constrained. It is shown that there exists a two dimensional projection including the center vector but with
this vector no longer located in the center of the two dimensional cone in the projection. This demonstrates that deviations in some directions are more constrained compared to others.

The size of the largest angle between the center of an assurance cone and the different extreme rays is analyzed for increasing number of inputs/outputs in Appendix. The difference between the largest and the smallest angle between the center and the extreme rays is shown to approach zero for increasing dimension with the rate of convergence in some cases slow. It is also shown in Appendix that the avoidance of implicit non-symmetric constraints of this type calls for non-polyhedral smooth assurance cones.
References:


APPENDIX.

1. Confidence intervals with equal probability mass on each side of the interval.

The certainty equivalent for (3) as

$$\frac{c_{ij}u_i - \bar{\epsilon}_i}{\sigma_i} \in [\eta_i^-, \eta_i^+]$$
$$\frac{c_{ij}u_j - \bar{\epsilon}_j}{\sigma_j} \in [\eta_j^-, \eta_j^+]$$

$$\forall (i, j) \in IJ$$, for some \( \alpha \in \mathbb{R}_+ \)

where \( \eta_k^- \equiv \Phi^{-1}(\alpha_k^-) \), \( \eta_k^+ \equiv \Phi^{-1}(\alpha_k^+) \), \( k = i, j \), and \( \Phi() \) is the distribution function for the distribution with zero mean and variances equal to one. (A1) can be rewritten as

$$\frac{\bar{\epsilon}_i + \eta_i^+ \sigma_i}{\bar{\epsilon}_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\bar{\epsilon}_i + \eta_i^- \sigma_i}{\bar{\epsilon}_j + \eta_j^+ \sigma_j}, \forall (i, j) \in IJ,$$

(A2)

In the general case with a symmetric distribution two sided confidence intervals are most often determined with equal probability mass on each side of the mean value. In Section 2 we assume that this is the case, i.e. \( 0.5 - \alpha_i^- = \alpha_i^+ - 0.5 \). For this case (A2) can be rewritten as in (2)

$$\frac{\bar{\epsilon}_i + \eta_i^+ \sigma_i}{\bar{\epsilon}_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\bar{\epsilon}_i + \eta_i^- \sigma_i}{\bar{\epsilon}_j + \eta_j^+ \sigma_j}, \forall (i, j) \in IJ,$$

(A3)

If, however, unequal probability mass is placed on each side of this vector then we have another situation. (A2) is in this case where \( \eta_k^+ \neq -\eta_k^- \) no longer a cone spanned by the Cartesian product of the partial confidence intervals. Furthermore, the cone determined from (A2) will be affected by the order in which the \( s \) prices is represented in the set \( IJ \). To see this consider the following two sets of constraints, where we in (A4) ((A5)) constrain \( u_i / u_j \) \((u_j / u_i)\):

$$\frac{f_{ij}^+}{f_{ij}^-} \equiv \frac{\bar{\epsilon}_i + \eta_i^+ \sigma_i}{\bar{\epsilon}_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\bar{\epsilon}_i + \eta_i^- \sigma_i}{\bar{\epsilon}_j + \eta_j^+ \sigma_j} \equiv \frac{f_{ij}^+}{f_{ij}^-}, \forall (i, j) \in IJ,$$

(A4)

$$\frac{f_{ij}^+}{f_{ij}^-} \equiv \frac{\bar{\epsilon}_j + \eta_j^+ \sigma_j}{\bar{\epsilon}_i + \eta_i^- \sigma_i} \geq \frac{u_i}{u_j} \geq \frac{\bar{\epsilon}_j + \eta_j^- \sigma_j}{\bar{\epsilon}_i + \eta_i^+ \sigma_i} \equiv \frac{f_{ij}^+}{f_{ij}^-}, \forall (i, j) \in IJ,$$

(A5)

Since, in general \( f_{ij}^+ \neq f_{ij}^- \) and \( f_{ij}^+ \neq f_{ij}^- \) we have that (A2) is not necessarily a cone spanned by the Cartesian product of the partial confidence intervals.

2. Confidence intervals with unequal probability mass on each side of the interval.
Theorem 1: Let $A$ and $B$ be

$$A \equiv \left\{ \mathbf{u} \in \mathbb{R}_+^s : \frac{\xi_i + \eta_i^+ \sigma_i}{\xi_j + \eta_j^- \sigma_j} \geq \frac{u_i}{u_j} \geq \frac{\xi_i + \eta_i^- \sigma_i}{\xi_j + \eta_j^+ \sigma_j}, \forall (i,j) \in IJ, \right\}$$

$$B \equiv \left\{ \mathbf{u} \in \mathbb{R}_+^s : \frac{c u_i - \xi_i}{\sigma_i} \in [\eta_i^-, \eta_i^+] \cap [\eta_j^-, \eta_j^+] \cap \left[ \frac{c u_j - \xi_j}{\sigma_j}, \frac{c u_j - \xi_j}{\sigma_j} \right], \forall (i,j) \in IJ, \text{ for some } c \in \mathbb{R}_+ \right\}$$

Then $A = B$.

Proof of Theorem 1:

Notice, that the set $B$ corresponds to (4) if we assume $c_{ij} = c, \forall (i,j) \in IJ,$ and if $\eta_i^- = \Phi^{-1} (\alpha_i), \eta_i^+ = - \Phi^{-1} (\alpha_i)$, for $\alpha_i \in (0,0.5)$. We will now show that $A = B$ i.e. there exists a common non-negative scaling factor $c$ such that $c u_i$ belongs to the confidence interval for the stochastic variable $e_i$, $i = 1, ..., s$. Clearly, $B \subseteq A$. To show that $B \supseteq A$ we consider a fixed $\hat{\mathbf{u}} \in A$, $\hat{u}_i > 0$, $i = 1, ..., s$. Let

$$\frac{c_i + \eta_i^- c_i}{\hat{u}_i} = \max_{i = 1, \ldots, s} \frac{c_i + \eta_i^+ c_i}{\hat{u}_i} \quad \text{and} \quad \frac{c_i + \eta_i^- c_i}{\hat{u}_i} = \min_{i = 1, \ldots, s} \frac{c_i + \eta_i^+ c_i}{\hat{u}_i} \quad (A6)$$

The set of feasible $c_{ij}, \forall (i,j) \in IJ$ for this fixed $\hat{\mathbf{u}} \in A$ is as follows

$$C \equiv \left\{ c_{ij}, (i,j) \in IJ : \frac{c_i + \eta_i^- c_i}{\hat{u}_i} \leq c_{ij} \leq \frac{c_i + \eta_i^+ c_i}{\hat{u}_i}, \frac{c_j + \eta_j^- c_j}{\hat{u}_j} \leq c_{ij} \leq \frac{c_j + \eta_j^+ c_j}{\hat{u}_j} \right\}$$

The set $D \equiv \left\{ c_{ij}, (i,j) \in IJ : \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h} \leq c_{ij} \leq \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h}, \frac{c_{ij} + \eta_{ij} c_{ij}}{\hat{u}_i} \leq \frac{c_{ij} + \eta_{ij} c_{ij}}{\hat{u}_i} \right\}$ is non-empty because $\hat{\mathbf{u}} \in A$, which implies that $\frac{c_{ih} + \eta_{ih} c_{ih}}{c_{ih} + \eta_{ih} c_{ih}} \leq \frac{\hat{u}_h}{\hat{u}_i} \leq \frac{c_{ij} + \eta_{ij} c_{ij}}{c_{ij} + \eta_{ij} c_{ij}} \leq \frac{\hat{u}_i}{\hat{u}_i}$. Hence we can choose in $D$ a common scalar

$$\hat{c} = c_{ij} \in \left[ \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h}, \frac{c_{ij} + \eta_{ij} c_{ij}}{\hat{u}_i} \right] \forall (i,j) \in IJ, D \subseteq C, \text{ hence we can find a common scalar } \hat{c} \text{ in the set } C \text{ of feasible } c_{ij}.$$ 

Let us choose $\hat{c} = \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h}$ and let us show that $\hat{\mathbf{u}} \in B$. Since $\hat{\mathbf{u}} \in A$, we have that

$$\frac{\left( \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h}, \frac{c_{ij} + \eta_{ij} c_{ij}}{\hat{u}_i} \right)}{\left( \frac{c_{ih} + \eta_{ih} c_{ih}}{\hat{u}_h}, \frac{c_{ij} + \eta_{ij} c_{ij}}{\hat{u}_i} \right)} \leq \frac{c_i + \eta_i^- c_i}{\sigma_i} \leq \frac{c_j + \eta_j^+ c_j}{\sigma_j}$$

($e_i + \eta_i^- \sigma_i \geq 0$ by assumption) and from (A6) that
\[
\left( \frac{e_{ii} + \eta^+_{ii} \sigma_{ii}}{\sigma_{ii} + \eta^-_{ii} \sigma_{ii}} \right) \frac{\tilde{\mu}_i}{\tilde{\mu}_0} \geq 1
\]

Hence,
\[
\left( \frac{e_{ii} + \eta^+_{ii} \sigma_{ii}}{\sigma_{ii} + \eta^-_{ii} \sigma_{ii}} \right) \frac{\tilde{\mu}_i}{\tilde{\mu}_0} \in \left[ 1, \frac{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}}{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}} \right] \text{ or } \left( e_{ii} + \eta^+_{ii} \sigma_{ii} \right) \frac{\tilde{\mu}_i}{\tilde{\mu}_0} \in \left[ \frac{\eta^-_{ii} \sigma_{ii}}{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}}, \frac{\eta^+_{ii} \sigma_{ii}}{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}} \right].
\]

or
\[
\left( \frac{e_{ii} + \eta^+_{ii} \sigma_{ii}}{\tilde{\mu}_0} \right) \frac{\tilde{\mu}_i}{\tilde{\mu}_0} \in \left[ \frac{\eta^-_{ii} \sigma_{ii}}{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}}, \frac{\eta^+_{ii} \sigma_{ii}}{\eta^-_{ii} \sigma_{ii} + \eta^+_{ii} \sigma_{ii}} \right].
\]

\[\blacksquare\]

3. **The size of the angle between a center vector in a symmetric AR and the extreme rays in the cone**

In section 4 we consider the feasible set of \( u_1, u_2 \) and \( u_3 \), given \( u_1 - u_2 = 0 \) in Figure 5. The angle \( \alpha \) between the mean vector and \( r_1 \) is larger than the angle \( \beta \) between \( r_2 \) and the mean vector and with an increasing number of outputs the difference between these angles will approach zero, but, as illustrated below, the convergence is slow if \( \alpha_{i,j} \) is small and/or \( \beta_{i,j} \) is large in the cone constraints in (1). We have the following result:

**Theorem 2.** The angles between the mean vector \((1, \ldots, 1) \in \mathbb{R}^s\) and all extreme rays in \( C_1((\beta_{i,j})^{-1}, \beta_{i,j}, (i,j) \in IJ) \subseteq \mathbb{R}^s\), \( \beta_{i,j} = \beta > 1 \), \((i,j) \in IJ\) fall between

- the largest angle: \( \arccos \left( \frac{1 + (s-1)\beta}{\sqrt{s^2\beta^2 + (s-1)^2}} \right) \), and
- the smallest angle: \( \arccos \left( \frac{\beta + (s-1)}{\sqrt{s^2\beta^2 + (s-1)^2}} \right) \).

**Proof:** We express \( C_1((\beta)^{-1}1, \beta 1) \) in intersection form as \( \{ u : Cu \geq 0, u \geq 0 \} \), \( C \) is a \((s(s-1)/2) \times n\) matrix. The \( i \)th row in \( C \) has two non-zero entries, \( s_i \) and \( t_i \), where \((s_i, t_i)\) is either \((\beta, -1)\) or a \((-1, \beta)\).

a) \([1, \beta, \ldots, \beta]\) is an extreme ray in \( C_1((\beta)^{-1}1, \beta 1) \), \( 1 \equiv [1, \ldots, 1] \) because we can find exactly \((s-1)\) linear independent constraint that are binding while the other \(s(s-1)/2 - (s-1)\) constraint are not violated. The \((s-1)\) binding constraints corresponds to the rows with a \( \beta \) in first position, i.e. \([\beta, -1, 0, \ldots, 0], [\beta, 0, -1, 0, \ldots, 0], \ldots, [\beta, 0, 0, \ldots, -1] \). Clearly, all other constraints are not violated.

b) \([1, 1, \ldots, 1, \beta]\) is an extreme ray in \( C_1((\beta)^{-1}1, \beta 1) \), \( 1 \equiv [1, \ldots, 1] \) because we can find exactly \((s-1)\) linear independent constraint that are binding while the other \(s(s-1)/2 - (s-1)\) constraint are not violated. The \((s-1)\) binding constraints corresponds
to the rows with a $-1$ in last position, i.e. $[\beta, 0, \ldots, 0, -1]$, $[0, \beta, 0, \ldots, -1], \ldots, [0, 0, \ldots, \beta, -1]$. Clearly, all other constraints are not violated.

The angle between $[1, \ldots, 1]$ and $[1, \ldots, 1, \beta]$ is

$$\arccos \left( \frac{1 + (s-1)\beta}{\sqrt{(s-1)\beta^2}} \right) = \arccos \left( \frac{1 + (s-1)\beta}{\sqrt{s\beta^2}} \right)$$

The angle between $[1, \ldots, 1]$ and $[1, 1, \ldots, 1, \beta]$ is

$$\arccos \left( \frac{\beta + (s-1)}{\sqrt{(s-1)\beta^2}} \right) = \arccos \left( \frac{\beta + (s-1)}{\sqrt{s\beta^2}} \right)$$

The rate of convergence for the largest and the smallest angle for $\beta \in \{2, 10, 100\}$ is illustrated in Figure A1. The figure illustrates that for a reasonable large number of output dimensions we still have a large gap between the largest and the smallest angle.

![Figure A1a](image_url)
Figure A1b. The angles between the vector [1, 100, 1] and the extreme rays [1, 100, 1] and [100, 1, 1] for increasing dimension.

4. The generation of smooth cones with a constant angle between a center vector in a symmetric AR and the (infinitely many) extreme rays in the cone.

**Theorem 3:** Denote the unit vectors in $\mathbb{R}^3$ rotated around the vector $(1,1,1)$ by $R(e_i, \gamma)$, $i = 1, 2, 3$. These vectors can be expressed as:

\[
R(e_1, \gamma) = \frac{1}{\gamma + 1 + 1}((1 + \gamma), \gamma(1 + \gamma), -\gamma), \\
R(e_2, \gamma) = \frac{1}{\gamma + 1 + 1}(-\gamma, (1 + \gamma), \gamma(1 + \gamma)) \quad \text{for } \gamma \in \mathbb{R}_+ \\
R(e_3, \gamma) = \frac{1}{\gamma + 1 + 1}(\gamma(1 + \gamma), -\gamma, (1 + \gamma))
\]

$R(e_i, 0) = e_i$ and $\lim_{\gamma \to \infty} R(e_i, \gamma) = e_{i+1}$, $i = 1, 2, 3$, $(e_4 = e_1)$. Hence, letting $\gamma$ go towards infinity will rotate the unit vectors around the vector $(1,1,1)$.

**Proof:**

Let us consider an arbitrary vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ which satisfies the following three conditions:
\[ \begin{align*}
\text{i)} & \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \{(x, y, z) \mid y = \gamma x, \text{ for some } \gamma \in [0, \infty)\} \\
\text{ii)} & \quad \text{angle} \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \text{angle} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \arccos \left( \frac{1}{\sqrt{3}} \right) \\
\text{iii)} & \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \\
\end{align*} \]

A vector satisfying i) and ii) has the following form \[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{(1 + \gamma)}{\sqrt{\gamma^2 + \gamma + 1}} \\ \frac{\gamma + \gamma(1 + \gamma) - \gamma^2}{\sqrt{\gamma^2 + \gamma + 1}} \\ \frac{-\gamma}{\sqrt{\gamma^2 + \gamma + 1}} \end{bmatrix},
\]
and solving by iii) for \(x\), we get \(\gamma^2 x^2 + x^2 + 1 + (1 + \gamma)^2 x^2 - 2(1 + \gamma)x = 1\) or \(x = \frac{1 + \gamma}{\gamma^2 + \gamma + 1}\). Hence,

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{(1 + \gamma)}{\sqrt{\gamma^2 + \gamma + 1}} \\ \frac{\gamma + \gamma(1 + \gamma) - \gamma^2}{\sqrt{\gamma^2 + \gamma + 1}} \\ \frac{-\gamma}{\sqrt{\gamma^2 + \gamma + 1}} \end{bmatrix}
\]

\[
R(e_i, \gamma)^R(e_j, \gamma) = 0, \ (i, j) = (1, 2), (1, 3), (2, 3):
\]
i, \(j = 1, 2 : (\gamma^2 + \gamma + 1)^2 R(e_i, \gamma)^R(e_j, \gamma) = (1 + \gamma)(-\gamma + \gamma(1 + \gamma) - \gamma^2) = 0
\]
i, \(j = 1, 3 : (\gamma^2 + \gamma + 1)^2 R(e_i, \gamma)^R(e_j, \gamma) = (1 + \gamma)(\gamma(1 + \gamma) - \gamma^2 - \gamma) = 0
\]
i, \(j = 2, 3 : (\gamma^2 + \gamma + 1)^2 R(e_i, \gamma)^R(e_j, \gamma) = (1 + \gamma)(-\gamma^2 - \gamma + \gamma(1 + \gamma)) = 0
\]

\[
\| R(e_i, \gamma) \| = \frac{1}{\sqrt{\gamma^2 + \gamma + 1}} \sqrt{(\gamma^2 + \gamma + 1)^2 + \gamma^2 + (1 + \gamma)^2} = \frac{1}{\sqrt{\gamma^2 + \gamma + 1}} \sqrt{\gamma^4 + 2\gamma^3 + 3\gamma^2 + 2\gamma + 1} = 1
\]

\[
R(e_i, \gamma)(1, 1, 1) = R(e_j, \gamma)(1, 1, 1), \ (i, j) = (1, 2), (1, 3), (2, 3).
\]

Angle \(\theta\) between \(R(e_i, \gamma)\) and \((1, 1, 1)\), \(i = 1, 2, 3:\)

\[
\cos \theta = \frac{1}{(\gamma^2 + \gamma + 1)\sqrt{3}} \left[ (1 + \gamma, \gamma(1 + \gamma), -\gamma)[(1, 1, 1)] \right] = \frac{(\gamma^2 + \gamma + 1)}{(\gamma^2 + \gamma + 1)\sqrt{3}} = (\sqrt{3})^{-1}
\]

Remark: Consider \(\gamma = 1\), i.e.
and let us consider the assurance region $C_1\left(\frac{1}{2}, 2\right)$ expressed in the basis $R(e_i, 1)$, $i = 1, 2, 3$.

We get the following six inequalities:

\[
\begin{align*}
+1 & \quad R(e_2, 1)^\top u - 2 R(e_1, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} -5 & -2 & 4 \end{pmatrix}^\top u \leq 0 \\
-2 & \quad R(e_2, 1)^\top u + 1 R(e_1, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} 4 & -2 & -5 \end{pmatrix}^\top u \leq 0 \\
+1 & \quad R(e_3, 1)^\top u - 2 R(e_1, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} -2 & 4 & -5 \end{pmatrix}^\top u \leq 0 \\
-2 & \quad R(e_3, 1)^\top u + 1 R(e_1, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} -5 & 4 & -2 \end{pmatrix}^\top u \leq 0 \\
+1 & \quad R(e_3, 1)^\top u - 2 R(e_2, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} 4 & -5 & -2 \end{pmatrix}^\top u \leq 0 \\
-2 & \quad R(e_3, 1)^\top u + 1 R(e_2, 1)^\top u \leq 0 \quad \text{or} \quad \begin{pmatrix} 4, 4, 7 \end{pmatrix} \\
7 & \quad 4, 7, 4 \\
5 & \quad 5, 2 \\
2 & \quad 2, 5, 5 \\
5 & \quad 5, 2, 5 \\
\end{align*}
\]

The six extreme rays generated from these six halfspaces are

\[
angle((1, 1, 1), (4, 4, 7)) = angle((1, 1, 1), (2, 2, 1)) = \arccos\left(\frac{5}{3\sqrt{3}}\right)
\]

\[
angle((1, 1, 1), (5, 5, 2)) = angle((1, 1, 1), (1, 1, 2)) = \arccos\left(\frac{4}{\sqrt{6}3}\right)
\]

**Lemma 1:** For any $\alpha \in [0, 2\pi)$ there exist $\gamma \in [0, \infty)$ such that the vector $\frac{1}{2}e_j - e_i$, $(i, j) = (1, 2), (1, 3), (2, 3)$ in $\mathbb{R}^3$ rotated $\alpha$ around $(1, 1, 1)^\top$ can be expressed as \[\left[\frac{1}{2}R(e_j, \gamma) - R(e_i, \gamma)\right].\]

**Proof:** W.l.o.g. let $j = 2$, $i = 1$. Let $[x, y, z]^\top$ in $\mathbb{R}^3$ be an arbitrary vector that we get if we make a rotation of $\alpha$ of the vector $\frac{1}{2}e_2 - e_1 = (\begin{pmatrix} -1, 1, 0 \end{pmatrix}^\top$ around $(1, 1, 1)^\top$. $[x, y, z]^\top$ can be expressed as the difference between two vectors $[x_1, y_1, z_1]^\top$ and $[x_2, y_2, z_2]^\top$, where $[x_1, y_1, z_1]^\top$ is $\frac{1}{2}e_2$ rotated $\alpha$ around $(1, 1, 1)^\top$ and $[x_2, y_2, z_2]^\top$ is the vector $e_1$ rotated $\alpha$ around $(1, 1, 1)^\top$. Hence, from theorem 3 we know that there exist $\gamma \in [0, \infty)$ such that

\[
[x_1, y_1, z_1]^\top = \frac{1}{2}R(e_2, \gamma), \quad [x_2, y_2, z_2]^\top = R(e_1, \gamma)
\]

\[
[x, y, z]^\top = [x_2, y_2, z_2]^\top - [x_2, y_2, z_2]^\top = \left[\frac{1}{2}R(e_2, \gamma) - R(e_1, \gamma)\right]
\]

\]
Remark: In particular we have

\[
\begin{align*}
\frac{1}{2} R(e_2, 0) - R(e_1, 0) &= [-1, \frac{1}{2}, 0] \\
\frac{1}{2} R(e_2, 1) - R(e_1, 1) &= \frac{1}{3} [-2\frac{1}{2}, -1, 2] \\
\frac{1}{2} R(e_2, \infty) - R(e_1, \infty) &= [0, -1, \frac{1}{2}]
\end{align*}
\]

Lemma 2. The set \(\tilde{C}^2(\frac{1}{2}, 2)\) defined as

\[
\tilde{C}^2(\frac{1}{2}, 2) = \left\{ u \mid \frac{1}{2} \leq \frac{R(e_i, \gamma)^T u}{R(e_j, \gamma)^T u} \leq 2, (i, j) = (1, 2), (1, 3), (2, 3), \gamma \in \mathbb{R}_+ \right\}
\]

can be expressed as

\[
\tilde{C}^2(\frac{1}{2}, 2) = \left\{ u \mid \text{angle}((1, 1, 1)^T, u) \leq \frac{\pi}{2} - \arccos \frac{1}{\sqrt{3}} \right\}
\]

Proof: Consider the cone \(\tilde{C}^2(\frac{1}{2}, 2)\) that emerges when we impose the same constraints as in \(C(\frac{1}{2}[1,1,1], 2[1,1,1])\) on prices expressed by all rotated bases:

\[
\tilde{C}^2(\frac{1}{2}, 2) = \left\{ u \mid \frac{1}{2} \leq \frac{R(e_i, \gamma)^T u}{R(e_j, \gamma)^T u} \leq 2, (i, j) = (1, 2), (1, 3), (2, 3), \gamma \in \mathbb{R}_+ \right\} =
\]

\[
\left\{ u \mid \begin{align*}
\frac{1}{2} R(e_2, \gamma) - R(e_1, \gamma)^T u &\leq 0, \\
-2 R(e_2, \gamma) + R(e_1, \gamma)^T u &\leq 0, \\
\frac{1}{2} R(e_3, \gamma) - R(e_1, \gamma)^T u &\leq 0, \\
-2 R(e_3, \gamma) + R(e_1, \gamma)^T u &\leq 0, \\
\frac{1}{2} R(e_2, \gamma) - R(e_3, \gamma)^T u &\leq 0, \\
-2 R(e_2, \gamma) + R(e_3, \gamma)^T u &\leq 0,
\end{align*} \quad \gamma \in \mathbb{R}_+ \right\}
\]

and consider the polar cone \(\tilde{C}^\circ(\frac{1}{2}, 2)\) to \(\tilde{C}^2(\frac{1}{2}, 2)\):

\[
\tilde{C}^\circ(\frac{1}{2}, 2) = \left\{ u \mid u = t_1 \left[ \frac{1}{2} R(e_2, \gamma) - R(e_1, \gamma) \right] + t_2 \left[ -2 R(e_2, \gamma) + R(e_1, \gamma) \right] + t_3 \left[ \frac{1}{2} R(e_3, \gamma) - R(e_1, \gamma) \right] + t_4 \left[ -2 R(e_3, \gamma) + R(e_1, \gamma) \right] + t_5 \left[ \frac{1}{2} R(e_2, \gamma) - R(e_3, \gamma) \right] + t_6 \left[ -2 R(e_2, \gamma) + R(e_3, \gamma) \right], \quad t_i \in \mathbb{R}_+, \quad i = 1, \ldots, 6, \quad \gamma \in \mathbb{R}_+ \right\}
\]
The six rays $\frac{1}{2}R(e_j, \gamma) - R(e_i, \gamma)$, $(i, j) = (1, 2), (1, 3), (2, 3)$ and $-2R(e_j, \gamma) + R(e_i, \gamma)$, $(i, j) = (1, 2), (1, 3), (2, 3)$ are extreme rays in this polar cone and the angles between these 6 rays and the mean vector $(-1, -1, -1)^t$ are equal to $\arccos \frac{1}{\sqrt{3}\sqrt{5}}$

$$\text{angle} \left( -\left[ \frac{1}{2}R(e_j, \gamma) - R(e_i, \gamma) \right], (1, 1, 1)^t \right) =$$
$$\arccos \left( \frac{0.5 \times (\gamma^2 + \gamma + 1)}{(\gamma^2 + \gamma + 1) \sqrt{3} \sqrt{5}} \right) = \arccos \frac{1}{\sqrt{3\sqrt{5}}} \approx 1.309, \ (i, j) = (1, 2), (1, 3), (2, 3) \text{ and}$$

$$\text{angle} \left( -\left[ -2R(e_j, \gamma) + R(e_i, \gamma) \right], (1, 1, 1)^t \right) =$$
$$\arccos \left( \frac{(\gamma^2 + \gamma + 1)}{(\gamma^2 + \gamma + 1) \sqrt{3\sqrt{5}}} \right) = \arccos \frac{1}{\sqrt{3\sqrt{5}}} \approx 1.309, \ (i, j) = (1, 2), (1, 3), (2, 3).$$

Hence, $-\mathcal{C}_1^o \left( \frac{1}{2}, 2 \right) = \left\{ u \mid \text{angle}((1, 1, 1)^t, u) \leq \arccos \frac{1}{\sqrt{3\sqrt{5}}} \right\}$

and

$$\mathcal{C}_1 \left( \frac{1}{2}, 2 \right) = \left\{ u \mid \text{angle}((1, 1, 1)^t, u) \leq \frac{\pi}{2} - \arccos \frac{1}{\sqrt{3\sqrt{5}}} \right\}.$$